Deterministic Income under a Stochastic Interest Rate

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1. Classical Problem: Maximizing Discounted Dividends in a Brownian Risk Model

2. Maximizing Discounted Consumption under a Deterministic Income
We consider an insurance company whose wealth is modelled by a Brownian motion with drift:

\[ X_t = x + \mu t + \sigma W_t , \quad t \geq 0 . \]

- \( x \): the initial capital
- \( \mu \in \mathbb{R} \): the drift parameter
- \( \sigma \): the diffusion parameter
- \( W \): a standard Brownian motion on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\), adapted to \( \{\mathcal{F}_t\} \).
Classical Problem: Maximizing Discounted Dividends in a Brownian Risk Model

Dividends

A dividend strategy \( \{C_t\} \) models the accumulated dividend payments up to time \( t \).

The Ex-dividend process is

\[
X_t^C = X_t - C_t, \quad t \geq 0
\]

and time to ruin under \( C \)

\[
\tau^C := \inf\{ t : X_t^C \leq 0 \}.
\]

A dividend strategy \( C \) is **admissible** if

- \( C \) is adapted, right-continuous and non-decreasing;
- \( X_t^C \geq 0 \) for all \( t \leq \tau^C \).
Classical Problem: Maximizing Discounted Dividends in a Brownian Risk Model

Discounting

Let $\delta > 0$ denote a fixed discounting rate.

**The target**

is to maximise the expected discounted dividend payments until ruin:

$$\max_{C} \mathbb{E} \left[ \int_{0}^{\tau_C} e^{-\delta t} \, dC_t \right],$$

to find the optimal strategy $C^*$ and the corresponding value function $V(x) := \mathbb{E} \left[ \int_{0}^{\tau_C} e^{-\delta t} \, dC_t^* \right]$. 
Classical Problem: Maximizing Discounted Dividends in a Brownian Risk Model

The Optimal Strategy

It turns out that the optimal strategy is a barrier strategy.

- Assume $C_t = \int_0^t c_s \, ds$ with $c_s \in [0, \xi]$ and $\xi > 0$. Then, there is an $x_0 \in \mathbb{R}_+$ such that the optimal strategy is given by

$$C_t^* = \xi \mathbb{1}_{[X_t^* > x_0]}.$$ 

- If we allow for lump sum payments, the optimal strategy becomes

$$C_t^* = \max \{ \sup_{0 \leq s \leq \tau_{C^*}} X_s - x_0, 0 \}.$$
A constant discount rate simplifies the calculations;

- The economic growth is unlikely to be constant over long time horizons.

- A constant discount rate assumption does not adequately account for uncertainty.
1. Classical Problem: Maximizing Discounted Dividends in a Brownian Risk Model

2. Maximizing Discounted Consumption under a Deterministic Income
   - Geometric Brownian Motion as an Interest Rate Process
   - Discounting by the Price of a Pure-Discount Bond at Time Zero
   - Ornstein-Uhlenbeck Process as an Interest Rate
Classical Problem: Maximizing Discounted Dividends in a Brownian Risk Model

Maximizing Discounted Consumption under a Deterministic Income

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Consider an individual or household, whose wealth is modelled by a deterministic process:

\[ X_t = x + \mu t \quad t \geq 0. \]

Define further

\[ r_t = r + mt + \sigma W_t, \]

where \( \{W_t\} \) is a standard Brownian motion and \( m > \frac{\sigma^2}{2} \).

We target to maximize the discounted consumption and define the return function corresponding to a strategy \( C = \{c_s\} \) and the value function to be

\[
V^C(r, x) = \mathbb{E}\left[ \int_0^\infty e^{-rs} c_s \, ds \mid r_0 = r \right] \quad (r, x) \in \mathbb{R} \times \mathbb{R}_+,
\]

\[
V(r, x) = \sup_C V^C(r, x) \quad (r, x) \in \mathbb{R} \times \mathbb{R}_+.
\]
Maximizing Discounted Consumption under a Deterministic Income, Geometric Brownian Motion as an Interest Rate Process

HJB Equation

A strategy $C = \{c_s\}$ is called admissible if

- $c_s \in [0, \xi]$, $\xi > 0$,
- $X_t^C \geq 0$ for all $t \in \mathbb{R}_+$,
- $C$ is adapted and cadlag.

The HJB equation corresponding to the problem is

$$\mu V_x + m V_r + \frac{\sigma^2}{2} V_{rr} + \sup_{0 \leq c \leq \xi} c \{e^{-r} - V_x\} = 0.$$
Maximizing Discounted Consumption under a Deterministic Income, Geometric Brownian Motion as an Interest Rate Process

The Optimal Strategy

Consider the strategy \( \tilde{C} = \{\tilde{c}_s\} \) for \( \xi > \mu \)

\[
\tilde{c}_s = \begin{cases} 
\xi & 0 \leq s \leq \frac{x}{\xi - \mu} \\
\mu & s > \frac{x}{\xi - \mu}
\end{cases}
\]

The corresponding return function is given by

\[
V^{\tilde{C}}(r, x) = \xi \int_{0}^{\frac{x}{\xi - \mu}} e^{-r - \frac{\sigma^2}{2} s} \, ds + \mu \int_{\frac{x}{\xi - \mu}}^{\infty} e^{-r - \frac{\sigma^2}{2} s} \, ds.
\]

and

\[
V^{\tilde{C}}(r, x) = e^{-r - \frac{\sigma^2}{2} \frac{x}{\xi - \mu}}.
\]

Obviously, this function solves the HJB equation.
If we allow for lump sum payments, the optimal strategy is to pay out the initial capital immediately and to pay on the rate $\mu$ up to an infinite time horizon. The value function is then given by

$$V(r, x) = e^{-r} x + e^{-r} \frac{\mu}{m - \frac{\sigma^2}{2}}.$$
1 Classical Problem: Maximizing Discounted Dividends in a Brownian Risk Model

2 Maximizing Discounted Consumption under a Deterministic Income

- Geometric Brownian Motion as an Interest Rate Process
- Discounting by the Price of a Pure-Discount Bond at Time Zero
- Ornstein-Uhlenbeck Process as an Interest Rate Process
Let \( \{r_s\} \) be an Ornstein-Uhlenbeck process. I.e. \( \{r_s\} \) fulfills the following integral equation

\[
    r_t = re^{-at} + \tilde{b}(1 - e^{-at}) + \tilde{\sigma}e^{-at} \int_0^t e^{as} \, dW_s,
\]

where \( r \) is the initial value of the process, \( a, \tilde{\sigma} > 0, \tilde{b} \in \mathbb{R} \) are constants and \( \{W_s\} \) is a standard Brownian motion. Further

\[
    U^r_s = \int_0^s r_u \, du \quad \text{if} \quad r_0 = r.
\]

Here, \( E[e^{-U^r_s}] \) denotes the price at zero of a zero-coupon bond (or pure-discount bond) with maturity \( s \).
Maximizing Discounted Consumption under a Deterministic Income, Discounting by the Price of a Pure-Discount Bond at Time Zero

HJB Equation

The return function corresponding to an admissible consumption rate strategy \( C = \{c_s\} \) (\( c_s \in [0, \xi] \) and \( X_t^C \geq 0 \) for \( t \in [0, T] \)) is defined as

\[
V^C(t, x) = \int_t^T \mathbb{E}\left[e^{-U_r s}c_s\right] ds + X_T^C \mathbb{E}\left[e^{-U_T}\right],
\]

We target to maximize the value of discounted consumption.

\[
V(t, x) = \sup_C V^C(t, x).
\]

The HJB equation corresponding to the problem is given by

\[
V_t + \mu V_x + \sup_{0 \leq c \leq \xi} c \left\{ \mathbb{E} \left[ e^{-U_t} \right] - V_x \right\} = 0.
\]
Let $\sigma := \tilde{\sigma} \sqrt{2a}$ and $b := \tilde{b} - \frac{\sigma^2}{2a^2}$. Then,

$$\mathbb{E}[e^{-U_s}] = \exp \left\{ -bs - \frac{r - b}{a} (1 - e^{-as}) - \frac{\sigma^2}{a^2} (1 - e^{-as})^2 \right\}.$$ 

Let

$$f(s) := -bs - \frac{r - b}{a} (1 - e^{-as}) - \frac{\sigma^2}{a^2} (1 - e^{-as})^2.$$ 

Then, the HJB equation becomes

$$V_t + \mu V_x + \sup_{0 \leq c \leq \xi} c \{ e^{f(t)} - V_x \} = 0.$$
Depending on the parameter choice, the function $f(s)$ will have different properties.
The idea is to establish a backward algorithm by cutting the function $f(t)$ in different areas.

Starting at $t = T$, we find the optimal strategy until the next change point in the behaviour of the function $f$.

Here, one can also consider an arbitrary drift function $\mu(t)$ with just finitely many zeros in the interval $[0, T]$. Then, one has to take into consideration the behaviour of the function $\mu(t)$.

This algorithm can be applied to an arbitrary deterministic discounting function, for example on $\sin(t)$.

The case of unrestricted payments is very easy. Basically, one has to wait until a local maximum and pay out everything there.
\[ f(t + \frac{x}{\xi - \mu}) \]

\[ f(t + \tilde{h}(t, x)) \]

\[ f(t + \tilde{h}(t, x) + \frac{x + \mu \cdot \tilde{h}(t, x)}{\xi - \mu}) \]

\[ f(t) \]

\[ f(h_3(f(t))) \]
Maximizing Discounted Consumption under a Deterministic Income, Discounting by the Price of a Pure-Discount Bond at Time Zero

Example
1. Classical Problem: Maximizing Discounted Dividends in a Brownian Risk Model

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\[ r_t = re^{-at} + \tilde{b}(1 - e^{-at}) + \tilde{\sigma}e^{-at} \int_0^t e^{as} \, dW_s, \]

Here, we assume that the long-term mean \( \tilde{b} \) of the process \( \{r_s\} \) fulfills: \( \tilde{b} > \frac{\tilde{\sigma}^2}{2a^2} \).

The return function corresponding to a strategy \( C = \{c_s\} \) and the value function are given by

\[
V^C(r, x) = \mathbb{E} \left[ \int_0^\infty e^{-U^r_s} c_s \, ds | X_0 = x \right], \quad (r, x) \in \mathbb{R} \times \mathbb{R}_+, \\
V(r, x) = \sup_C V^C(r, x), \quad (r, x) \in \mathbb{R} \times \mathbb{R}_+. 
\]
Maximizing Discounted Consumption under a Deterministic Income, Ornstein-Uhlenbeck Process as an Interest Rate

Realisations of \( \exp\{-U^r_s\} \), \( r = 1, \tilde{b} = 4, \tilde{b} = 0, \tilde{b} = -1 \)
Maximizing Discounted Consumption under a Deterministic Income, Ornstein-Uhlenbeck Process as an Interest Rate

The HJB Equation

\[
\mu V_x + a(\tilde{b} - r)V_r + \frac{\tilde{\sigma}^2}{2} V_{rr} - rV + \sup_{0 \leq c \leq \xi} c \left\{ 1 - V_x \right\} = 0 .
\]
Maximizing Discounted Consumption under a Deterministic Income, Ornstein-Uhlenbeck Process as an Interest Rate
Restricted Rates with $\xi \leq \mu$

The return function $V^\xi$ corresponding to the constant strategy $c_s \equiv \xi$ is:

$$V^\xi(r, x) = \xi \mathbb{E} \left[ \int_0^\infty e^{-Us} \, ds \right] = \xi \int_0^\infty e^{f(r, s)} \, ds .$$

Note that $V^\xi$ does not depend on $x$ in this case. In particular:
$$1 - V^\xi_x(r, x) = 1.$$ And it is an easy exercise to prove that $V^\xi$ solves the ODE

$$a(\tilde{b} - r)v_r + \frac{\tilde{\sigma}^2}{2}v_{rr} - rv + \xi = 0 .$$

$\Rightarrow V^\xi$ is the value function.
Maximizing Discounted Consumption under a Deterministic Income, Ornstein-Uhlenbeck Process as an Interest Rate

Restricted rates with $\xi > \mu$.

The return function corresponding to the strategy

$$\hat{c}_s = \begin{cases} \xi & 0 \leq s \leq \frac{x}{\xi-\mu} \\ \mu & s > \frac{x}{\xi-\mu} \end{cases}$$

is given by

$$V(\hat{c}(r,x)) = E \left[ \xi \int_0^{\frac{x}{\xi-\mu}} e^{-Us} ds + \mu \int_{\frac{x}{\xi-\mu}}^{\infty} e^{-Us} ds \right]$$

$$= \xi \int_0^{\frac{x}{\xi-\mu}} e^{f(r,s)} ds + \mu \int_{\frac{x}{\xi-\mu}}^{\infty} e^{f(r,s)} ds.$$
Maximizing Discounted Consumption under a Deterministic Income, Ornstein-Uhlenbeck Process as an Interest Rate

Restricted rates with $\xi > \mu$. II

The derivative of $V^\hat{C}$ with respect to $x$ is given by

$$V_x^\hat{C}(r, x) = e^{f(r, \frac{x}{\xi - \mu})}.$$ 

If $r < 0$ and $s > 0$, then for every fixed $r \in \mathbb{R}$ the function $f(r, s)$ is at first increasing and then decreasing in $s$. Further, since $f(r, 0) = 0$ for all $r \in \mathbb{R}$ and $\lim_{s \to \infty} f(r, s) = -\infty$ the curve

$$\alpha(s) := \frac{a}{1 - e^{-as}} \left\{ -bs + \frac{b}{a}(1 - e^{-as}) - \frac{\sigma^2}{2a^2}(1 - e^{-as})^2 \right\}$$

is unique with $f(\alpha(s), s) \equiv 0$. $\therefore V^\xi$ is not the value function.
Maximizing Discounted Consumption under a Deterministic Income, Ornstein-Uhlenbeck Process as an Interest Rate

Restricted rates with \( \xi > \mu \). III

Let now

\[
\tau := \inf \{ t \geq 0 : r_t = 0, \ r_0 = r < 0 \} \\
\varrho := \inf \{ t \geq 0 : r_t = 0, \ r_0 = r > 0 \}
\]

and

\[
G(r, x) := \mathbb{E} \left[ e^{-U^r_{\tau}} (x + \mu \tau + C) \right] \\
F(r, x) := \mathbb{E} \left[ \xi \int_0^{x/(\xi - \mu)^\wedge \varrho} e^{-U^r_s} \ ds + \mu \int_{(x/(\xi - \mu)^\wedge \varrho)}^{\varrho} e^{-U^r_s} \ ds \right. \\
\left. + e^{-U^r_\varrho} \left( x + (\mu - \xi) \left( \frac{x}{\xi - \mu} \wedge \varrho \right) + C \right) \right]
\]
Maximizing Discounted Consumption under a Deterministic Income, Ornstein-Uhlenbeck Process as an Interest Rate

Restricted rates with $\xi > \mu$. IV

Then: $G(0, x) = F(0, x)$ and $G_x(0, x) = F_x(0, x) = 1$ for all $x \geq 0$.

$$G_x(r, x) = \mathbb{E}\left[e^{-Ur}x^{\xi-\mu}^\varrho\right] > 1 \quad \text{for } r < 0, \ x \in \mathbb{R}_+ \text{ and }$$

$$F_x(r, x) = \mathbb{E}\left[e^{-Ur/\xi}x^{\xi-\mu}^\varrho\right] < 1 \quad \text{for } r > 0, \ x \in \mathbb{R}_+.$$
Maximizing Discounted Consumption under a Deterministic Income, Ornstein-Uhlenbeck Process as an Interest Rate

Unrestricted Consumption

HJB

$$\max \left\{ \mu V_x + a(\tilde{b} - r)V_r + \frac{\tilde{\sigma}^2}{2} V_{rr} - rV, 1 - V_x \right\} = 0.$$ 

Using the same notation like above, we let

$$G(r, x) = \mathbb{E}\left[ (x + \mu \tau + G(0, 0)) e^{-U_r} \right] \quad \text{for } r \leq 0,$$

$$F(r, x) = x + \mathbb{E}\left[ \mu \int_0^\tau e^{-U_s} \, ds + G(0, 0) e^{-U_\theta} \right] \quad \text{for } r \geq 0.$$ 

Obviously, $G_x(r, x) = \mathbb{E}[e^{-U_r}] > 1$ for $r < 0$ and $F_x(r, x) = 1$. Then, it is clear $G(0, x) = F(0, x)$ and $G_x(0, x) = F_x(0, x)$. 

