Deterministic Income under a Stochastic Interest Rate

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Agenda

The Surplus Process

We consider an insurance company whose wealth is modelled by a Brownian motion with drift:

$$X_t = x + \mu t + \sigma W_t \ , \quad t \geq 0 \ .$$

- x: the initial capital
- $\mu \in \mathbb{R}$: the drift parameter
- σ: the diffusion parameter
- W: a standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, adapted to $\{\mathcal{F}_t\}$.

Dividends

A dividend strategy $\{C_t\}$ models the accumulated dividend payments up to time t.

The Ex-dividend process is

$$X_t^C = X_t - C_t \,, \quad t \geq 0$$

and time to ruin under C

$$\tau^C := \inf\{t : X_t^C \leq 0\}.$$

A dividend strategy C is admissible if

- C is adapted, right-continuous and non-decreasing;
- $X_t^C \geq 0$ for all $t \leq \tau^C$.

Discounting

Let $\delta > 0$ denote a fixed discounting rate.

The target

is to maximise the expected discounted dividend payments until ruin:

$$\max_{C} \mathbb{E}\left[\int_0^{\tau^C} e^{-\delta t} \, \mathrm{d}C_t \right] ,$$

to find the optimal strategy C^* and the corresponding value function $V(x) := \mathbb{E}\left[\int_0^{\tau^C} e^{-\delta t} \, \mathrm{d}C_t^* \right]$.

The Optimal Strategy

It turns out that the optimal strategy is a barrier strategy.

- Assume $C_t = \int_0^t c_s \, \mathrm{d}s$ with $c_s \in [0, \xi]$ and $\xi > 0$. Then, there is an $x_0 \in \mathbb{R}_+$ such that the optimal strategy is given by

$$c_t^* = \xi \mathbb{1}_{[X_t^{C^*} > x_0]} \, .$$

- If we allow for lump sum payments, the optimal strategy becomes

$$C_t^* = \max\{ \sup_{0 \le s \le \tau^{C^*}} X_s - x_0, 0\} \, .$$

Constant Discount Rate?

- + A constant discount rate simplifies the calculations;
- − The economic growth is unlikely to be constant over long time horizons.
- − A constant discount rate assumption does not adequately account for uncertainty.

1. Classical Problem: Maximizing Discounted Dividends in a Brownian Risk Model

2. Maximizing Discounted Consumption under a Deterministic Income
   - Geometric Brownian Motion as an Interest Rate Process
   - Discounting by the Price of a Pure-Discount Bond at Time Zero
   - Ornstein-Uhlenbeck Process as an Interest Rate

1. Classical Problem: Maximizing Discounted Dividends in a Brownian Risk Model

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Consider an individual or household, whose wealth is modelled by a deterministic process:

$$X_t = x + \mu t \quad t \geq 0 \ .$$

Define further

$$r_t = r + mt + \sigma W_t,$$

where $\{W_t\}$ is a standard Brownian motion and $m > \frac{\sigma^2}{2}$.

We target to maximize the discounted consumption and define the return function corresponding to a strategy $C = \{c_s\}$ and the value function to be

$$V^C(r,x) = \mathbb{E}\Big[\int_0^\infty e^{-r_s} c_s \, \mathrm{d}s | r_0 = r\Big] \quad (r,x) \in \mathbb{R} \times \mathbb{R}_+ \ ,$$
$$V(r,x) = \sup_C V^C(r,x) \quad (r,x) \in \mathbb{R} \times \mathbb{R}_+ \ .$$

HJB Equation

A strategy $C = \{c_s\}$ is called admissible if

- $c_s \in [0, \xi]$, $\xi > 0$,
- $X_t^C \geq 0$ for all $t \in \mathbb{R}_+$,
- C is adapted and cadlag.

The HJB equation corresponding to the problem is

$$\mu V_x + m V_r + \frac{\sigma^2}{2} V_{rr} + \sup_{0 \leq c \leq \xi} c \left\{ e^{-r} - V_x \right\} = 0 \,.$$

The Optimal Strategy

Consider the strategy $\hat{C} = \{\hat{c}_s\}$ for $\xi > \mu$

$$\hat{c}_s = \begin{cases} \xi & 0 \leq s \leq \frac{x}{\xi-\mu} \\ \mu & s > \frac{x}{\xi-\mu} \end{cases} .$$

The corresponding return function is given by

$$V^{\hat{C}}(r,x) = \xi \int_0^{\frac{x}{\xi-\mu}} e^{-r-(m-\frac{\sigma^2}{2})s} \, \mathrm{d}s + \mu \int_{\frac{x}{\xi-\mu}}^{\infty} e^{-r-(m-\frac{\sigma^2}{2})s} \, \mathrm{d}s .$$

and

$$V_x^{\hat{C}}(r,x) = e^{-r-(m-\frac{\sigma^2}{2})\frac{x}{\xi-\mu}} .$$

Obviously, this function solves the HJB equation.

Unrestricted Payments

If we allow for lump sum payments, the optimal strategy is to pay out the initial capital immediately and to pay on the rate μ up to an infinite time horizon. The value function is then given by

$$V(r,x) = e^{-r}x + e^{-r}\frac{\mu}{m - \frac{\sigma^2}{2}} \ .$$

Let $\{r_s\}$ be an Ornstein-Uhlenbeck process . I.e. $\{r_s\}$ fulfils the following integral equation

$$r_t = re^{-at} + \tilde{b}(1 - e^{-at}) + \tilde{\sigma}e^{-at} \int_0^t e^{as} \, \mathrm{d}W_s \; ,$$

where r is the initial value of the process, $a, \tilde{\sigma} > 0$, $\tilde{b} \in \mathbb{R}$ are constants and $\{W_s\}$ is a standard Brownian motion.
Further

$$U_s^r = \int_0^s r_u \, \mathrm{d}u \quad \text{if } r_0 = r.$$

Here, $\mathbb{E}\left[e^{-U_s^r}\right]$ denotes the price at zero of a zero-coupon bond (or pure-discount bond) with maturity s.

HJB Equation

The return function corresponding to an admissible consumption rate strategy $C = \{c_s\}$ ($c_s \in [0, \xi]$ and $X_t^C \geq 0$ for $t \in [0, T]$) is defined as

$$V^C(t,x) = \int_t^T \mathbb{E}\left[e^{-U_s^r}\right] c_s \, \mathrm{d}s + X_T^C \mathbb{E}\left[e^{-U_T^r}\right],$$

We target to maximize the value of discounted consumption.

$$V(t,x) = \sup_C V^C(t,x) .$$

The HJB equation corresponding to the problem is given by

HJB

$$\boldsymbol{V_t + \mu V_x + \sup_{0 \leq c \leq \xi} c\left\{\mathbb{E}\left[e^{-U_t^r}\right] - V_x\right\} = 0 .}$$

Pure-Discount Bond

Let $\sigma := \tilde{\sigma}\sqrt{2a}$ and $b := \tilde{b} - \frac{\sigma^2}{2a^2}$. Then,

$$\mathbb{E}[e^{-U_s^r}] = \exp\left\{ -bs - \frac{r-b}{a}\left(1-e^{-as}\right) - \frac{\sigma^2}{a^2}\left(1-e^{-as}\right)^2 \right\}.$$

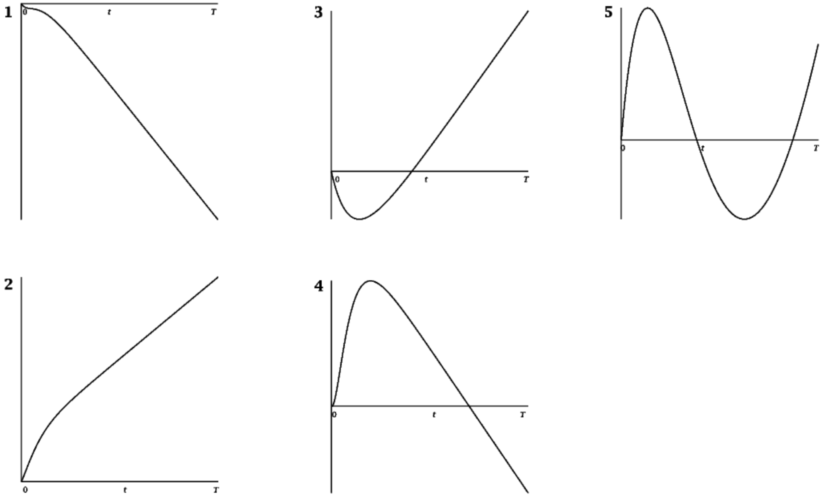
Let

$$\boldsymbol{f(s) := -bs - \frac{r-b}{a}\left(1-e^{-as}\right) - \frac{\sigma^2}{a^2}\left(1-e^{-as}\right)^2}.$$

Then, the HJB equation becomes

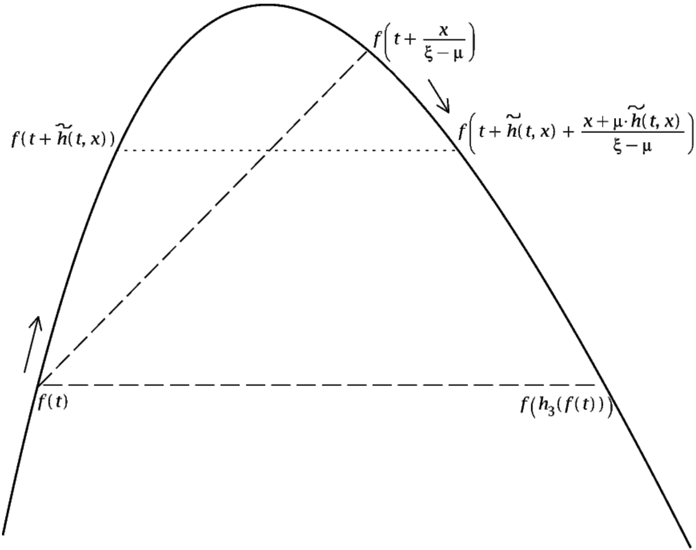
$$V_t + \mu V_x + \sup_{0 \le c \le \xi} c\left\{e^{f(t)} - V_x\right\} = 0.$$

Depending on the parameter choice, the function $f(s)$ will have different properties.



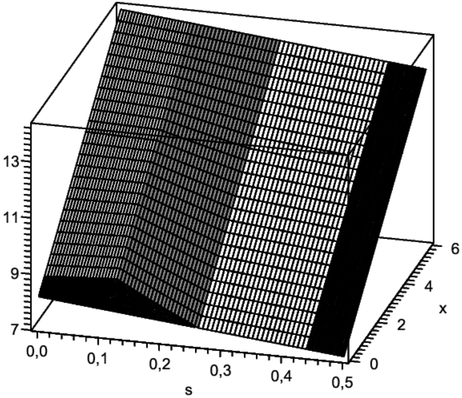
The Value Function

- The idea is to establish a backward algorithm by cutting the function $f(t)$ in different areas.
- Starting at $t = T$, we find the optimal strategy until the next change point in the behaviour of the function f.
- Here, one can also consider an arbitrary drift function $\mu(t)$ with just finitely many zeros in the interval $[0, T]$. Then, one has to take into consideration the behaviour of the function $\mu(t)$.
- This algorithm can be applied to an arbitrary deterministic discounting function, for example on $\sin(t)$.
- The case of unrestricted payments is very easy. Basically, one has to wait until a local maximum and pay out everything there.



$f\left(t+\frac{x}{\xi-\mu}\right)$
$f(t+\widetilde{h}(t,x))$
$f\left(t+\widetilde{h}(t,x)+\frac{x+\mu\cdot\widetilde{h}(t,x)}{\xi-\mu}\right)$
$f(t)$
$f(h_3(f(t)))$

Example



Maximizing Discounted Consumption under a Deterministic Income, Ornstein-Uhlenbeck Process as an Interest Rate

$$r_t = re^{-at} + \tilde{b}(1 - e^{-at}) + \tilde{\sigma} e^{-at} \int_0^t e^{as} \, \mathrm{d}W_s \ ,$$

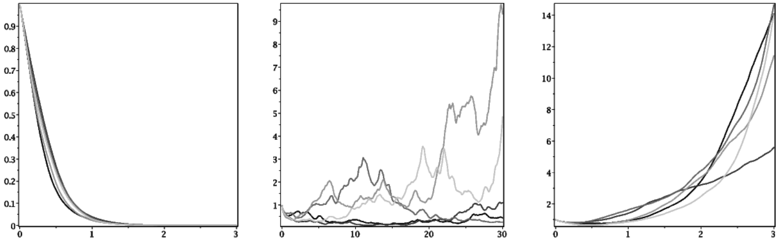
Here, we assume that the long-term mean $\tilde{b}$ of the process $\{r_s\}$ fulfils: $\tilde{b} > \frac{\tilde{\sigma}^2}{2a^2}$.

The return function corresponding to a strategy $C = \{c_s\}$ and the value function are given by

$$V^C(r, x) = \mathbb{E}\Big[\int_0^{\infty} e^{-U_s^r} c_s \, \mathrm{d}s | X_0 = x\Big], \quad (r, x) \in \mathbb{R} \times \mathbb{R}_+ \ ,$$

$$V(r, x) = \sup_C V^C(r, x), \quad (r, x) \in \mathbb{R} \times \mathbb{R}_+ \ .$$

Realisations of $\exp\{-U_s^r\}$, $r = 1, \tilde{b} = 4, \tilde{b} = 0, \tilde{b} = -1$



The HJB Equation

HJB

$$\mu V_x + a(\tilde{b} - r)V_r + \frac{\tilde{\sigma}^2}{2} V_{rr} - rV + \sup_{0 \le c \le \xi} c\left\{1 - V_x\right\} = 0 \, .$$

Restricted Rates with $\xi \leq \mu$

The return function V^ξ corresponding to the constant strategy $c_s \equiv \xi$ is:

$$V^\xi(r,x) = \xi \mathbb{E}\Big[\int_0^\infty e^{-U_s^r}\,\mathrm{d}s\Big] = \xi \int_0^\infty e^{f(r,s)}\,\mathrm{d}s\,.$$

Note that V^ξ does not depend on x in this case. In particular: $1 - V_x^\xi(r,x) = 1$. And it is an easy exercise to prove that V^ξ solves the ODE

$$a(\tilde{b} - r)v_r + \frac{\tilde{\sigma}^2}{2}v_{rr} - rv + \xi = 0\,.$$

$\rightsquigarrow$ V^ξ is the value function.

Restricted rates with $\xi > \mu$. I

The return function corresponding to the strategy

$$\hat{c}_s = \begin{cases} \xi & 0 \le s \le \frac{x}{\xi-\mu} \\ \mu & s > \frac{x}{\xi-\mu} \end{cases}$$

is given by

$$V^{\hat{C}}(r,x) = \mathbb{E}\left[\xi \int_0^{\frac{x}{\xi-\mu}} e^{-U_s^r} \, \mathrm{d}s + \mu \int_{\frac{x}{\xi-\mu}}^{\infty} e^{-U_s^r} \, \mathrm{d}s\right]$$

$$= \xi \int_0^{\frac{x}{\xi-\mu}} e^{f(r,s)} \, \mathrm{d}s + \mu \int_{\frac{x}{\xi-\mu}}^{\infty} e^{f(r,s)} \, \mathrm{d}s.$$

Restricted rates with $\xi > \mu$. II

The derivative of $V^{\hat{C}}$ with respect to x is given by

$$V_x^{\hat{C}}(r,x) = e^{f\left(r, \frac{x}{\xi-\mu}\right)} .$$

If $r < 0$ and $s > 0$, then for every fixed $r \in \mathbb{R}_-$ the function $f(r,s)$ is at first increasing and then decreasing in s. Further, since $f(r,0) = 0$ for all $r \in \mathbb{R}$ and $\lim_{s\to\infty} f(r,s) = -\infty$ the curve

$$\alpha(s) := \frac{a}{1-e^{-as}}\left\{ -bs + \frac{b}{a}(1-e^{-as}) - \frac{\sigma^2}{2a^2}(1-e^{-as})^2 \right\}$$

is unique with $f\big(\alpha(s), s\big) \equiv 0$. $\rightsquigarrow$ V^{ξ} is not the value function.

Restricted rates with $\xi > \mu$. III

Let now

$$\tau := \inf\{t \geq 0 : \; r_t = 0, \; r_0 = r < 0\}$$
$$\varrho := \inf\{t \geq 0 : \; r_t = 0, \; r_0 = r > 0\}$$

and

$$G(r,x) := \mathbb{E}\Big[e^{-U^r_\tau}(x + \mu\tau + C)\Big]$$
$$F(r,x) := \mathbb{E}\Big[\xi \int_0^{\frac{x}{\xi-\mu}\wedge\varrho} e^{-U^r_s}\,\mathrm{d}s + \mu \int_{\frac{x}{\xi-\mu}\wedge\varrho}^{\varrho} e^{-U^r_s}\,\mathrm{d}s + e^{-U^r_\varrho}\Big(x + (\mu-\xi)\Big(\frac{x}{\xi-\mu}\wedge\varrho\Big) + C\Big)\Big]$$

Restricted rates with $\xi > \mu$. IV

Then: $G(0,x) = F(0,x)$ and $G_x(0,x) = F_x(0,x) = 1$ for all $x \geq 0$.

$$G_x(r,x) = \mathbb{E}\left[e^{-U^r_\tau}\right] > 1 \quad \text{for } r < 0,\ x \in \mathbb{R}_+ \text{ and}$$

$$F_x(r,x) = \mathbb{E}\left[e^{-U^r_{\frac{x}{\xi-\mu}\wedge\varrho}}\right] < 1 \quad \text{for } r > 0,\ x \in \mathbb{R}_+.$$

Unrestricted Consumption

HJB

$$\max\left\{\mu V_x + a(\tilde{b} - r)V_r + \frac{\tilde{\sigma}^2}{2} V_{rr} - rV, 1 - V_x\right\} = 0\,.$$

Using the same notation like above, we let

$$G(r,x) = \mathbb{E}\Big[\big(x + \mu\tau + G(0,0)\big)e^{-U_\tau^r}\Big] \quad \text{for } r \le 0,$$

$$F(r,x) = x + \mathbb{E}\Big[\mu \int_0^\varrho e^{-U_s^r}\,\mathrm{d}s + G(0,0)e^{-U_\varrho^r}\Big] \quad \text{for } r \ge 0\,.$$

Obviously, $G_x(r,x) = \mathbb{E}\big[e^{-U_\tau^r}\big] > 1$ for $r < 0$ and $F_x(r,x) = 1$.
Then, it is clear $G(0,x) = F(0,x)$ and $G_x(0,x) = F_x(0,x)$.

Literature

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