How Robust is the Value-at-Risk of Credit Risk Portfolios?

(joint work with L. Rüschendorf, S. Vanduffel, J. Yao)

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Agenda

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1. Background
   - Literature
   - Observations

Credit risk management

1. Management of credit risk is of utmost importance (Crisis 2008).
2. Portfolio models are subject to significant **model uncertainty** (defaults are **rare and correlated events**).
3. Recent studies (Embrechts et al. (2013,2014)) show that the impact of model uncertainty on Value-at-Risk (VaR) estimates is huge.

Credit risk management: Notation

- n individual risks $(L_1, L_2, ..., L_n)$ (risky loans)
- A portfolio $S := L_1 + ... + L_n$
- **Value-at-Risk** of S at level $q \in (0, 1)$

$$\text{VaR}_q(S) = F_S^{-1}(q) = \inf \{x \in \mathbb{R} \mid F_S(x) \geq q\}$$

Motivation on VaR aggregation

Full information on **marginal distributions**:
$L_j \sim F_j$ and represent risks as $L_j = F_j^{-1}(U_j)$
where U_j is $\mathcal{U}(0,1)$.

$$+$$

Full Information on **dependence**:
$(U_1, U_2, ..., U_n) \sim C$ (C is called the copula)

$$\Rightarrow$$

$\mathrm{VaR}_q(L_1 + L_2 + ... + L_n)$ can be computed!

Background,

Motivation on VaR aggregation

Full information on **marginal distributions**:
$L_j \sim F_j$ and represent risks as $L_j = F_j^{-1}(U_j)$
where U_j is $\mathcal{U}(0,1)$.

$$+$$

Partial or **no** Information on **dependence**:
$(U_1, U_2, ..., U_n) \sim ???$

$$\Rightarrow$$

$\mathrm{VaR}_q\,(L_1 + L_2 + ... + L_n)$ **cannot** be computed!

Only a range of possible values for $\mathrm{VaR}_q\,(L_1 + L_2 + ... + L_n)$.

Maximum VaR under Dependence Uncertainty

Bounds on Value-at-Risk

$$M := \sup VaR_q\left[L_1 + L_2 + \ldots + L_n\right],$$
subject to $L_j \sim F_j$, copula C = **unknown**

- **Explicit sharp bounds**

· $n = 2$ Makarov (1981), Rüschendorf (1982)
· homogeneous portfolios: Rüschendorf & Uckelmann (1991), Denuit, Genest & Marceau (1999), Embrechts & Puccetti (2006), Wang & Wang (2011), Bernard, Jiang and Wang (2014)
· heterogeneous portfolios: Wang & Wang (2015)

- **Approximate sharp bounds**

· The Rearrangement Algorithm (Puccetti & Rüschendorf (2012), Embrechts, Puccetti & Rüschendorf (2013))

Observations

- The **bound M may be too wide** to be practically useful:

a feature that can only be explained by the absence of dependence information.

- **Our objective: incorporate dependence information**

Bounds on Value-at-Risk

- VaR_q is **not** maximized for the comonotonic scenario:

$$S^c = L_1^c + L_2^c + ... + L_n^c$$

where all L_i^c are *comonotonic.*

$$\begin{aligned} M &\geq VaR_q \left[L_1^c + L_2^c + ... + L_n^c\right] \\ &= VaR_q \left[L_1\right] + VaR_q \left[L_2\right] + ... + VaR_q \left[L_n\right] \end{aligned}$$

where $(L_1^c, L_2^c, ...L_n^c)$ is a comonotonic copy of $(L_1, L_2, ...L_n)$, i.e.

$$(L_1^c, L_2^c, ...L_n^c) = (F_{L_1}^{-1}(U), F_{L_2}^{-1}(U), ..., F_{L_n}^{-1}(U)).$$

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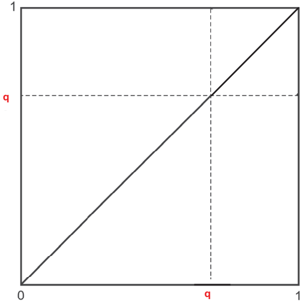
2 Unconstrained VaR Bounds

- VaR Bounds with 2 risks
- VaR Bounds with n risks
- Example

"Riskiest" Dependence: maximum VaR$_q$ in 2 dims

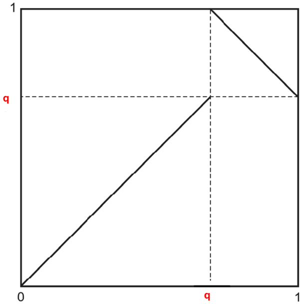
If L_1 and L_2 are U(0,1) comonotonic, then

$$VaR_q(S^c) = VaR_q(X_1) + VaR_q(X_2) = 2q.$$



"Riskiest" Dependence: maximum VaR_q in 2 dims

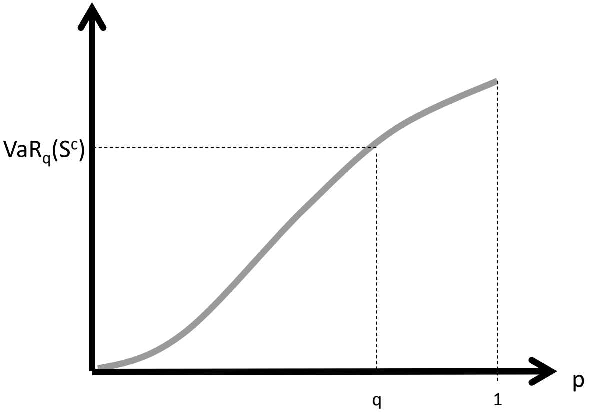
If L_1 and L_2 are U(0,1) and antimonotonic in the tail, then $VaR_q(S^*) = 1 + q$.



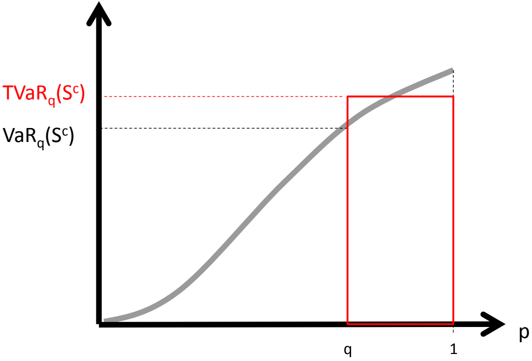
$$VaR_q(S^*) = 1 + q > VaR_q(S^c) = 2q$$

$\rightarrow$ to maximize VaR_q the idea is to change the comonotonic

VaR at level q of the comonotonic sum w.r.t. q

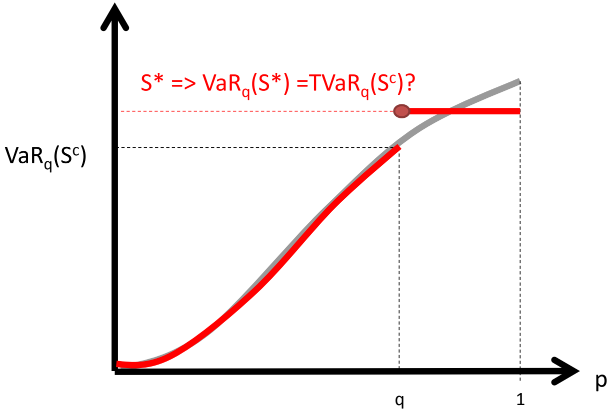


VaR at level q of the comonotonic sum w.r.t. q



where TVaR (Expected shortfall): $\mathrm{TVaR}_q(X) = \frac{1}{1-q}\int_q^1 \mathrm{VaR}_u(X)\mathrm{d}u$

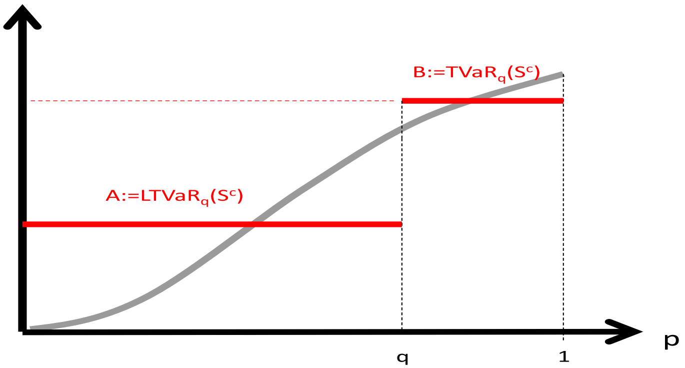
Riskiest Dependence Structure VaR at level q



Analytic expressions

Analytical Unconstrained Bounds with $L_j \sim F_j$

$$A = LTVaR_q(S^c) \leq \mathrm{VaR}_q\left[L_1 + L_2 + \ldots + L_n\right] \leq B = TVaR_q(S^c)$$



Proof for B

Upper bound for VaR with given marginals

$$\mathrm{VaR}_q\left[X_1 + X_2 + \ldots + X_n\right] \leq B := \mathrm{TVaR}_q\left[X_1^c + X_2^c + \ldots + X_n^c\right]$$

Here $(X_1^c, X_2^c, \ldots X_n^c)$ is a comonotonic copy of $(X_1, X_2, \ldots X_n)$, i.e.

$$(X_1^c, X_2^c, \ldots X_n^c) = (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)).$$

Proof:

$$\begin{aligned} \mathrm{VaR}_q\left[X_1 + X_2 + \ldots + X_n\right] &\leq \mathrm{TVaR}_q\left[X_1 + X_2 + \ldots + X_n\right] \\ &\leq \mathrm{TVaR}_q\left[X_1^c + X_2^c + \ldots + X_n^c\right] \end{aligned}$$

Illustration for the maximum VaR (1/3)

q				
1-q	8	0	3	Sum= 11
	10	1	4	Sum= 15
	11	7	7	Sum= 25
	12	8	9	Sum= 29

Illustration for the maximum VaR (2/3)

q				
1-q	**8**	**0**	**3**	Sum= 11
	10	**1**	**4**	Sum= 15
	11	**7**	**7**	Sum= 25
	12	**8**	**9**	Sum= 29

Rearrange **within** columns..to make the sums as constant as possible…

B=(11+15+25+29)/4=20

Illustration for the maximum VaR (3/3)

q				
1-q	8	8	4	Sum= 20
	10	7	3	Sum= 20
	12	1	7	Sum= 20
	11	0	9	Sum= 20

=B!

Agenda

3 VaR Bounds with Dependence Information

- Literature
- Problem

Constrained Problem

Finding minimum and maximum possible values for VaR of the credit portfolio loss, $L = \sum_{i=1}^{n} L_i$, given that

- we know the marginal distributions of the risks L_i.
- we have some **dependence information.**

Example 1: variance constraint - Bernard, Rüchendorf and Vanduffel (2015)

$$M := \sup \mathrm{VaR}_q \left[L_1 + L_2 + ... + L_n \right],$$
$$\text{subject to } \; L_j \sim F_j, \mathrm{var}(L_1 + L_2 + ... + L_n) \leq s^2$$

Example 2: VaR bounds when the joint distribution of $(L_1, L_2, ..., L_n)$ is known on a subset of the sample space: Bernard and Vanduffel (2015).

Description

It appears that adding dependence information can sharpen the bounds considerably. Here,

- VaR bounds with higher order moments on the portfolio sum
  - Portfolio loss

$$L = \sum_{i=1}^{n} L_i \text{ where } L_i \sim v_i B(p_i) \ (v_i \geq 0)$$

    Hence, L_i is a scaled Bernoulli rv.
  - We are interested in the problem:

$$M := \sup \mathrm{VaR}_q[L]$$
$$\text{subject to } L_i \sim v_i B(p_i) \text{ and } E[L^k] \leq c_k \ (k = 2, 3, ..., K).$$

- Extended version of the RA
- Assess model risk of industry credit risk models for VaR

VaR bounds with moment constraints

- Without moment constraints, VaR bounds are attained if there exists a dependence among risks L_i such that

$$L = \begin{cases} A & \text{probability } q \\ B & \text{probability } 1-q \end{cases} \quad \text{a.s.}$$

- If the "distance" between A and B is too wide then improved bounds are obtained with

$$L^* = \begin{cases} a & \text{with probability } q \\ b & \text{with probability } 1-q \end{cases}$$

such that

$$\begin{cases} a^k q + b^k(1-q) \leq c_k \\ aq + b(1-q) = E[L] \end{cases}$$

in which a and b are "as distant as possible while satisfying the constraint"

Dealing with moment constraints

To find a and b, solve for each $k = 2, 3, .., K$ the system of equations $(A \leq B)$

$$\begin{cases} Aq + B(1-q) = E(L) \\ A^k q + B^k (1-q) = c_k \end{cases}$$

and obtain $K - 1$ pairs $\{A_j, B_j\}$. Then, take

$$\begin{aligned} b &= \min \{B_j | j = 2, 3, ..., K\} \\ a &= \frac{E[L] - b(1-q)}{q}. \end{aligned}$$

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4. Approximate VaR Bounds
   - Rearrangement Algorithm
   - Standard Rearrangement Algorithm
   - Extended Rearrangement Algorithm

Approximating Sharp Bounds

- The bounds a and b are sharp if one can construct dependence among the risks L_i such that quantile function of their sum L becomes flat on $[0, q]$ and on $[q, 1]$. This holds true under certain conditions (see eg Wang and Wang, 2014).
- To approximate sharp VaR bounds: Extended Rearrangement Algorithm (RA).

Standard RA (Puccetti and Rüschendorf, 2012):

- Put the margins in a matrix
- Rearrange each column (adapt the dependence) such that L (row-sums) approximates a constant ($E[L]$)

Example

$N = 4$ observations of $d = 3$ variables: L_1, L_2, L_3

$$\mathbf{M} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{6} & \mathbf{3} \\ \mathbf{4} & \mathbf{0} & \mathbf{0} \\ \mathbf{6} & \mathbf{3} & \mathbf{4} \end{bmatrix}$$

Each column: **marginal** distribution
Interaction among columns: **dependence** among the risks

Standard RA: Sum with Minimum Variance

minimum variance with $d = 2$ risks L_1 and L_2

Antimonotonicity: $var(\mathbf{L_1^a} + L_2) \leq var(\mathbf{L_1} + L_2)$

Aggregate Risk with Minimum Variance

- Columns of M are rearranged such that they become anti-monotonic with the sum of all other columns.

$$\forall k \in \{1, 2, ..., d\}, \mathbf{L_k^a} \textit{ antimonotonic with } \sum_{j \neq k} L_j$$

- After each step, $var\left(\mathbf{L_k^a} + \sum_{j\neq k} L_j\right) \leq var\left(\mathbf{L_k} + \sum_{j\neq k} L_j\right)$ where $\mathbf{L_k^a}$ is antimonotonic with $\sum_{j\neq k} L_j$

Aggregate risk with minimum variance
Step 1: First column

$$\begin{array}{c} \downarrow \\ \begin{bmatrix} 6 & 6 & 4 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \begin{array}{c} X_2 + X_3 \\ 10 \\ 5 \\ 2 \\ 0 \end{array} \quad \text{becomes} \quad \begin{bmatrix} 0 & 6 & 4 \\ 1 & 3 & 2 \\ 4 & 1 & 1 \\ 6 & 0 & 0 \end{bmatrix}$$

Aggregate risk with minimum variance

$$\begin{bmatrix} \downarrow \\ \mathbf{6} & 6 & 4 \\ \mathbf{4} & 3 & 2 \\ \mathbf{1} & 1 & 1 \\ \mathbf{0} & 0 & 0 \end{bmatrix} \quad \begin{matrix} X_2 + X_3 \\ 10 \\ 5 \\ 2 \\ 0 \end{matrix} \quad \text{becomes} \quad \begin{bmatrix} \mathbf{0} & 6 & 4 \\ \mathbf{1} & 3 & 2 \\ \mathbf{4} & 1 & 1 \\ \mathbf{6} & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} & \downarrow & \\ 0 & \mathbf{6} & 4 \\ 1 & \mathbf{3} & 2 \\ 4 & \mathbf{1} & 1 \\ 6 & \mathbf{0} & 0 \end{bmatrix} \quad \begin{matrix} X_1 + X_3 \\ 4 \\ 3 \\ 5 \\ 6 \end{matrix} \quad \text{becomes} \quad \begin{bmatrix} 0 & \mathbf{3} & 4 \\ 1 & \mathbf{6} & 2 \\ 4 & \mathbf{1} & 1 \\ 6 & \mathbf{0} & 0 \end{bmatrix}$$

$$\begin{bmatrix} & & \downarrow \\ 0 & 3 & \mathbf{4} \\ 1 & 6 & \mathbf{2} \\ 4 & 1 & \mathbf{1} \\ 6 & 0 & \mathbf{0} \end{bmatrix} \quad \begin{matrix} X_1 + X_2 \\ 3 \\ 7 \\ 5 \\ 6 \end{matrix} \quad \text{becomes} \quad \begin{bmatrix} 0 & 3 & \mathbf{4} \\ 1 & 6 & \mathbf{0} \\ 4 & 1 & \mathbf{2} \\ 6 & 0 & \mathbf{1} \end{bmatrix}$$

Aggregate risk with minimum variance

Each column is antimonotonic with the sum of the others:

$$\begin{bmatrix} \downarrow & & \\ 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \begin{matrix} X_2+X_3 \\ 7 \\ 6 \\ 3 \\ 1 \end{matrix}, \quad \begin{bmatrix} & \downarrow & \\ 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \begin{matrix} X_1+X_3 \\ 4 \\ 1 \\ 6 \\ 7 \end{matrix}, \quad \begin{bmatrix} & & \downarrow \\ 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \begin{matrix} X_1+X_2 \\ 3 \\ 7 \\ 5 \\ 6 \end{matrix}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{6} & \mathbf{0} \\ \mathbf{4} & \mathbf{1} & \mathbf{2} \\ \mathbf{6} & \mathbf{0} & \mathbf{1} \end{bmatrix} \qquad S_N = \overset{X_1+X_2+X_3}{\begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \end{bmatrix}}$$

Illustration

Extended RA

q	...	...	...	-a
	...	...	...	-a
	...	...	...	-a
	...	...	...	-a
1-q	8	8	4	-b
	10	7	3	-b
	12	1	7	-b
	11	0	9	-b

Rearrange now within all columns such that all sums becomes close to zero

Extended RA

- ERA: Apply RA on the new matrix and check:
  – If all constraints are satisfied, then L^* readily generates the approximate solutions to the problem
  – If not, decrease b by ε, and compute a such as the expectation of L is satisfied. Apply the extended RA again.

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Corporate portfolio

- a corporate portfolio of a major European Bank.
- 4495 loans mainly to medium sized and large corporate clients
- total exposure (EAD) is 18642.7 (million Euros), and the top 10% of the portfolio (in terms of EAD) accounts for 70.1% of it.
- portfolio exhibits some heterogeneity.

Summary statistics of a corporate portfolio

	Minimum	Maximum	Average
Default probability	0.0001	0.15	0.0119
EAD	0	750.2	116.7
LGD	0	0.90	0.41

Comparison of Industry Models

VaRs of a corporate portfolio under different industry models

	$q =$	Comon.	KMV	Credit Risk$^+$	Beta
	95%	393.5	281.3	281.8	282.5
	95%	393.5	340.6	346.2	347.4
$\rho = 0.10$	99%	2374.1	539.4	513.4	520.2
	99.5%	5088.5	631.5	582.9	593.5

VaR bounds

With $\rho = 0.1$,

VaR assessment of a corporate portfolio

$q =$	KMV	Comon.	Unconstrained	$K = 2$	$K = 3$	$K = 4$
95%	340.6	393.3	(34.0 ; 2083.3)	(97.3 ; 614.8)	(100.9 ; 562.8)	(100.9 ; 560.6)
99%	539.4	2374.1	(56.5 ; 6973.1)	(111.8 ; 1245.0)	(115.0 ; 941.2)	(115.9 ; 834.7)
99.5%	631.5	5088.5	(89.4 ; 10119.9)	(114.9 ; 1709.4)	(117.6 ; 1177.8)	(118.5 ; 989.5)
99.9%	862.4	12905.1	(111.8 ; 14784.9)	(119.2 ; 3692.3)	(120.8 ; 1995.9)	(121.2 ; 1472.7)

- Obs 1: Comparison with analytical bounds
- Obs 2: Significant bounds reduction with moments information

Conclusions

1. We propose simple bounds for VaR of a portfolio when there is information on the higher order moments of the portfolio sum.
2. We propose a new algorithm to approximate sharp VaR bounds.
3. Considering additional moment constraints can strengthen the unconstrained VaR bounds significantly.
4. Illustration with credit risk models

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