Dynamic Portfolio Optimization with Bounded Shortfall Risks

Ralf Wunderlich

Brandenburg University of Technology Cottbus – Senftenberg

Joint work with	Jörn Sass	(Kaiserslautern)
	Rüdiger Frey	(Vienna, Austria)
	Abdelali Gabih	(Marrakech, Morocco)

Scientific Day of the German Actuarial Society

Bonn, April 30, 2014

Dynamic Portfolio Optimization

Financial market	containing risky and risk-free assets
	continuously tradable
Initial capital	$x_0 > 0$
Horizon	$[0, T]$
Aim	maximize expected utility of terminal wealth
	constrain the risk of falling short a benchmark
Problem	find an optimal investment strategy
	How many shares
	of **which** asset
	have to be held **at which time** **?**

Classical Black-Scholes Model

Bond $dB_t = rB_t\,dt, \quad B_0 = 1 \quad \Longrightarrow$ Bond price $\quad B_t = e^{rt}$

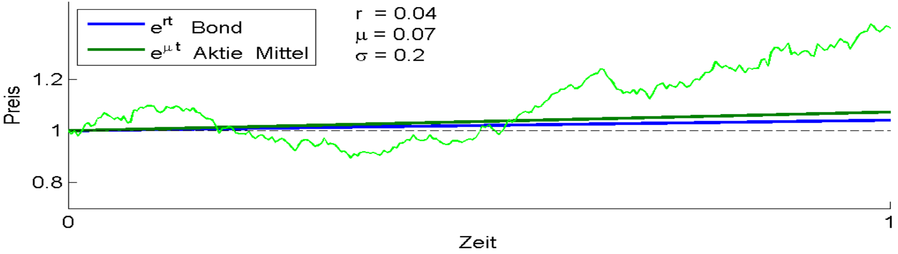
r risk-free interest rate (continuously compounded)

Stock $dS_t = S_t(\mu\,dt + \sigma\,dW_t), \quad S_0 = 1$

Stock price $\quad S_t = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right), \quad t \in [0, T]$

W_t Wiener process,

μ average stock return, drift; $\quad \sigma$ volatility



Portfolio

Initial capital $X_0 = x_0 > 0$

Wealth at time t invested in

$$X_t = \underbrace{\pi_t^0}_{\text{bond}} + \underbrace{\pi_t^1}_{\text{stock 1}} + \ldots + \underbrace{\pi_t^n}_{\text{stock n}}$$

$$\pi_t^k = \phi_t^k S_t^k = \text{number of shares} \times \text{price}, \ k = 1, \ldots n$$

Strategy $\boldsymbol{\pi_t = (\pi_t^1, \ldots, \pi_t^n)^\tau}$

Self financing condition (assume $r = 0$ for simplicity) $\Rightarrow$

Wealth equation

X_t satisfies **linear SDE** with initial value x_0

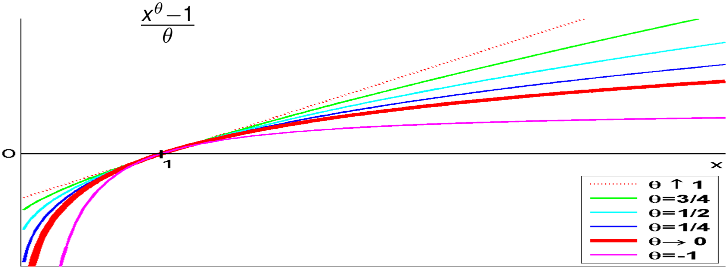
$$\begin{aligned} dX_t^\pi &= \pi_t^\tau \, (\mu \, dt + \sigma dW_t) \\ X_0^\pi &= x_0 \end{aligned}$$

Utility Function

$U : [0,\infty) \to \mathbb{R} \cup \{-\infty\}$ strictly increasing and concave

$$U(x) = \begin{cases} \frac{x^\theta}{\theta} & \text{für } \theta \in (-\infty, 1) \setminus \{0\} \quad \text{power utility} \\ \log x & \text{für } \theta = 0 \quad \text{log-utility} \end{cases}$$

CRRA - constant relative risk aversion: $-\frac{xU''(x)}{U'(x)} \equiv 1 - \theta$



Limiting cases $U(x) - \frac{1}{\theta} \underset{\theta \to 0}{\longrightarrow} \log x, \qquad U(x) \underset{\theta \uparrow 1}{\longrightarrow} x$

Classical Optimization Problem

Wealth equation $dX_t^\pi = \pi_t(\mu\, dt + \sigma dW_t), \quad X_0^\pi = x_0$

Strategy $\pi = (\pi_t)_{t\in[0,T]}$

Admissible Strategies $\mathcal{A}(x_0) = \{(\pi_t) : \text{integrability conditions}, \; X_t^\pi \geq 0,\ \forall t \in [0, T]\,\}$

Optimization problem

$$\max_{\pi\in\mathcal{A}(x_0)} \quad E\left[U(X_T^\pi)\right]$$

Solution optimal fraction of wealth $\dfrac{\pi_t^*}{X_t^*} = \dfrac{\mu - r}{(1-\theta)\sigma^2} = \text{const}$

MERTON (1969,1973), Nobel in prize economics 1997

using methods from dynamic programming

Solving Stochastic Optimal Control Problems

Optimization problem

$$\max_{\pi \in \mathcal{A}(x_0)} \quad E\left[U(X_T^\pi)\right]$$

- **Dynamic Programming**

  Hamilton-Jacobi-Bellman equation for the value function $V(t, x)$

- **Martingale Method**

  special method for solving portfolio optimization problems

  requires 'complete' financial market model

Drawbacks of the Merton Strategy

- Considerable **shortfall risks**

  High probabilities for small values of optimal terminal wealth

  $X_T^* < x_0$ initial capital

  $X_T^* < x_0 e^{rT}$ risk-free investment in a bond

  $\Longrightarrow$ **Optimization under bounded shortfall risk**

- **Drift μ is very difficult to estimate**

  need observations for very long time

  is not constant

  depends on the state of the economy

  $\Longrightarrow$ **Model with stochastic, non-observable drift μ**

Shortfall Risk

Compare terminal wealth X_T with benchmark q

e.g. $q \sim x_0$ initial capital

Shortfall if $X_T < q$

Risk Measure $\rho = E_Q\left[(X_T - q)^-\right]$ where $Q \sim P$

Expected Loss

Special Cases

- $Q = P$ **Future** Expected Loss (FEL)

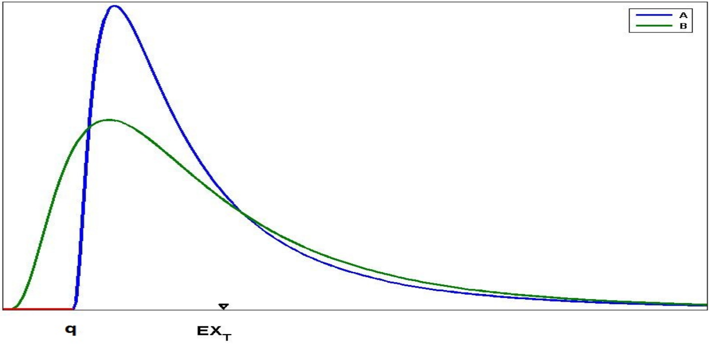
  $\rho = E\left[(X_T - q)^-\right]$
- $Q = \widetilde{P}$ martingale measure

  Present Expected Loss (PEL)

  $\rho = \widetilde{E}\left[(X_T - q)^-\right]$ option price

Shortfall Risk II

Variance is not a suitable measure for shortfall risk



mean and variance of X_T^A and X_T^B coincide

but $P(X_T^A < q) = \mathbf{0}$ while $P(X_T^B < q) = \alpha > \mathbf{0}$

Shortfall Risk III

Other risk measures

Expected Utility Loss $\quad \rho = E\left[(U(X_T) - U(q))^-\right]$

Shortfall prob. / Value at Risk (VaR) $\quad \rho = P(X_T < q)$

Conditional Value at Risk (CVaR)

Expected Shortfall

Coherent / convex risk measures, ...

Stochastic benchmark

e.g. $\quad q \sim X_T^{\overline{\pi}}$ index, reference portfolio

$q \sim S_T$ stock price, $n = 1$

Optimization Problem Under Risk Constraints

Wealth equation $dX_t^\pi = \pi_t^\tau (\mu_t \, dt + \sigma dW_t), \quad X_0^\pi = x_0$

Strategy $\pi = (\pi_t)_{t\in[0,T]}$

Admissible strategies $\mathcal{A}(x_0) = \{(\pi_t) :$ integrability conditions, $X_t^\pi \geq 0, \; \forall t \in [0, T] \}$

Optimization problem

$$\max_{\pi\in\mathcal{A}(x_0)} \quad E\left[U(X_T^\pi)\right]$$

risk constraint $\boldsymbol{E_Q\left[(X_T^\pi - q)^-\right] \leq \varepsilon}$

Martingale Method I

Probability space $(\Omega, \mathcal{F}, P)$

Filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ information structure

Stochastic process $(Z_t)_{t \in [0,T]}$ is called martingale, if

(Z_t) is adapted to $\mathbb{F}$, $E|Z_t| < \infty$ and

$$\boxed{E[Z_t | \mathcal{F}_s] = Z_s} \quad \text{for all } 0 \leq s \leq t \leq T$$

Properties mean $EZ_t = EZ_0 = \text{const}$

$E[Z_t - Z_s | \mathcal{F}_s] = 0$ martingales are driftless

$dZ_t = \mathbf{0}\, dt + \sigma_t dW_t$

Wiener process W_t and $e^{W_t - t/2}$ are martingales

stock price S_t, wealth X_t are not (in general)

Supermartingale $E[Z_t | \mathcal{F}_s] \leq Z_s$ for all $0 \leq s \leq t \leq T$

Martingale Method II

Martingale density

$$Z_t = \exp\Big(-\int_0^t \kappa_u^\tau \, dW_u - \tfrac{1}{2}\int_0^t \|\kappa_u\|^2 \, du \Big)$$

$$dZ_t = -\kappa_t^\tau Z_t \, dW_t, \quad Z_0 = 1$$

$$\kappa_t = \sigma^{-1}\mu_t \quad \text{market price of risk}$$

Assumptions to (μ_t) $\Rightarrow$ (Z_t) is $(\mathbb{F}, P)$-martingale

Martingale measure $\widetilde{P}(A) = E\left[Z_T 1_A\right]$ für $A \in \mathcal{F}_T$, $\widetilde{P} \sim P$

Assumptions to (π_t) $\Rightarrow$ (X_t^π) is $(\mathbb{F}, \widetilde{P})$-supermartingale

$$\Longrightarrow \quad \widetilde{E}\left[X_T^\pi\right] \le \widetilde{E}\left[X_0^\pi\right] = x_0 \quad \text{(budget constraint)}$$

$$\widetilde{W}_t = W_t + \int_0^t \kappa_u \, du \quad \text{is Wiener process w.r.t. } \widetilde{P}$$

Martingale representation theorem $\Rightarrow$ **completeness of market**

Let $\xi \ge 0$ be an $\mathcal{F}_T$-measurable r.v. with $\widetilde{E}[\xi] = x_0$.

Then there exists a strategy $(\pi_t) \in \mathcal{A}(x_0)$ such that $\xi = X_T^\pi$

Decomposition of the Optimization Problem

Dynamic problem $\max_{\pi \in \mathcal{A}(x_0)} E\left[U(X_T^\pi)\right]$ subject to $E_Q\left[(X_T^\pi - q)^-\right] \le \varepsilon$

Static problem $\max_{\xi \in \mathcal{B}(x_0)} E\left[U(\xi)\right]$ subject to $E_Q\left[(\xi - q)^-\right] \le \varepsilon$

where $\mathcal{B}(x_0) := \{\xi \ge 0 : \xi \text{ is } \mathcal{F}_T\text{-measurable}, \underbrace{\widetilde{E}[\xi] \le x_0}_{\text{budget-constraint}} \}$

terminal wealth generated from initial capital in $(0, x_0]$

$\rightarrow$ **optimal terminal wealth** $\xi^* = f(Z_T)$

Representation problem

find a strategy $\pi \in \mathcal{A}(x_0)$ such that $\xi^* = X_T^\pi$

$\rightarrow$ **optimal strategy** π_t^*

Static Problem

$$\max_{xi \in \mathcal{B}(x_0)} \quad E\left[U(\xi)\right]$$

$$\text{risk constraint} \qquad \boldsymbol{E_Q\left[(\xi - q)^-\right] \leq \varepsilon}$$

$$\mathcal{B}(x_0) := \{\xi \geq 0 : \xi \text{ is } \mathcal{F}_T\text{-measurable, } \widetilde{E}\left[\xi\right] \leq x_0\}$$

Choose the bound ε such that the risk constraint

- is binding $\quad \varepsilon \leq \varepsilon_{\text{max}} = E_Q\left[(X_T^M - q)^-\right]$

  risk of Merton portfolio (no constraint)

- can be fulfilled $\quad \varepsilon \geq \varepsilon_{\text{min}} = \begin{cases} 0 & \text{for } q \leq x_0 \text{ (portfolio insurer)} \\ \ldots > 0 & \text{for } q > x_0 \end{cases}$

  see GABIH, SASS, WUNDERLICH (2009)

Optimal Terminal Wealth

Theorem ($Q = \widetilde{P}$, Gabih, Sass, Wunderlich (2009), Basak, Shapiro (2001))

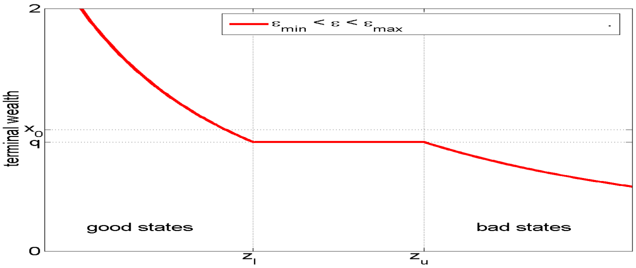
For $\varepsilon \in (\varepsilon_{\min}, \varepsilon_{\max})$ *the optimal terminal wealth is* $\xi^* = f(Z_T; y_1^*, y_2^*)$

$$\text{where} \quad f(z; y_1, y_2) \;=\; \begin{cases} I(y_1 z) & \text{for} \quad z \in (0, z_l] \\ q & \text{for} \quad z \in (z_l, z_u] \\ I(\,(y_1 - y_2) z) & \text{for} \quad z \in (z_u, \infty) \end{cases}$$

I *inverse of* U', $z_l = \frac{U'(q)}{y_1}$ *and* $z_u = \frac{U'(q)}{y_1 - y_2}$.

The real numbers $y_1^*, y_2^* > 0$ *uniquely solve the equations*

$$\widetilde{E}\,[f(Z_T; y_1, y_2)] = x_0 \qquad \text{and} \qquad E_Q\left[(f(Z_T; y_1, y_2) - q)^-\right] = \varepsilon.$$



Optimal Terminal Wealth

Theorem ($Q = \widetilde{P}$, GABIH, SASS, WUNDERLICH (2010), BASAK, SHAPIRO (2001))

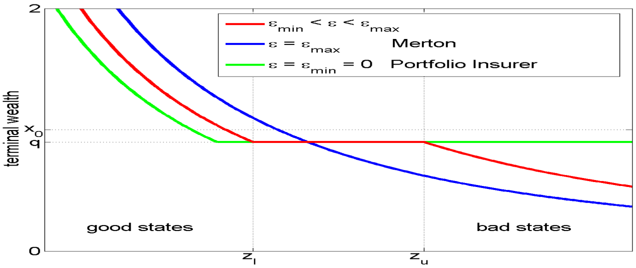
For $\varepsilon \in (\varepsilon_{\min}, \varepsilon_{\max})$ *the optimal terminal wealth is* $\xi^* = f(Z_T; y_1^*, y_2^*)$

$$\text{where} \quad f(z; y_1, y_2) = \begin{cases} I(y_1 z) & \text{for} \quad z \in (0, z_l] \\ q & \text{for} \quad z \in (z_l, z_u] \\ I(\,(y_1 - y_2) z) & \text{for} \quad z \in (z_u, \infty) \end{cases}$$

I *inverse of* U', $z_l = \frac{U'(q)}{y_1}$ *and* $z_u = \frac{U'(q)}{y_1 - y_2}$.

The real numbers $y_1^*, y_2^* > 0$ *uniquely solve the equations*

$$\widetilde{E}\,[f(Z_T; y_1, y_2)] = x_0 \qquad \text{and} \qquad E_Q\left[(f(Z_T; y_1, y_2) - q)^-\right] = \varepsilon.$$



Optimal Strategy

Clark formula

Let $D_t\xi$ be the Malliavin derivative of the $\mathcal{F}_T$-measurable r.v. $\xi \in D_{1,1}$, then it holds

$$\xi = \widetilde{E}[\xi] + \int_0^T \widetilde{E}\left[(D_t\xi)^\tau | \mathcal{F}_t\right] \, d\widetilde{W}_t$$

let $\xi = f(Z_T) = f(Z_T; y_1^*, y_2^*)$ (optimal terminal wealth)

prove chain rule for the Malliavin derivative $D_t\, f(Z_T) = f'(Z_T)\, D_t Z_T$

Wealth equation $\xi = x_0 + \int_0^T (\pi_t^*)^\tau \sigma \, d\widetilde{W}_t$

Comparison of coefficients $\Rightarrow$ optimal strategy is

$$\pi_t^* = \sigma^{-\tau}\, \widetilde{E}\left[f'(Z_T)\, D_t Z_T \,|\, \mathcal{F}_t\right]$$

Results for the Classical Black Scholes Model

(Z_t) is geometric Brownian motion and Z_T is lognormal

Explicit formulas for optimal terminal wealth $X_T^* = f(Z_T)$

optimal strategy $\pi_t^* = g(t, Z_t)$

Basak, Shapiro, Tepla (2001, 2002)	VaR, Present Expected Loss deterministic and stochastic benchmark
Gabih, Grecksch, Wunderlich (2004,...)	Future Expected Loss Expected Utility Loss deterministic and stochastic benchmark

Drawbacks of the Merton Strategy

✓ Considerable shortfall risks

$\Longrightarrow$ Optimization under bounded shortfall risk

- Drift μ is very difficult to estimate

  is not constant

$\Longrightarrow$ **model with stochastic, non-observable drift μ**

Models With Partial Information on the Drift I

Drift μ_t depends on an additional "source of randomness"

is not $\mathbb{F}^S$-adapted (not observable)

where $\mathbb{F}^S = (\mathcal{F}_t^S)_{t\in[0,T]} \subset \mathbb{F}$ filtration generated by (S_t)

Investor can only observe stock prices S_t

but neither the drift μ_t nor the Wiener process W_t

$\Longrightarrow$ model with **partial information**

Problem "learn" the drift from observed stock prices

find an estimate or **filter** for μ_t

Extension investor also receives extra information (expert opinions)

news, ratings, recommendations of analysts

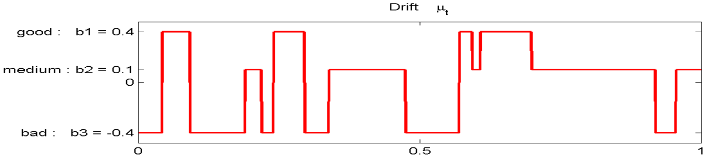
see Frey, Gabih, Wunderlich (2012, 2014)

see Gabih, Kondakji, Sass, Wunderlich (2014)

Models With Partial Information on the Drift II

Hidden Markov Model (HMM)

SASS, HAUSSMANN (2004), RIEDER, BÄUERLE (2005), NAGAI, RUNGGALDIER (2005)



Drift μ_t is a time-continuous homogeneous Markov chain
independent of W_t
switching between the states is controlled by intensity matrix

Linear Gaussian Model

LAKNER (1998), NAGAI, PENG (2002), BRENDLE (2006)

μ_t is a mean-reversion process

$d\mu_t = \alpha(\overline{\mu} - \mu_t)dt + \beta dW_t^1$, where W_t^1 is BM (in)dependent of W_t

HMM Filtering

Drift	$\boldsymbol{\mu_t}$	hidden state, signal
Returns	$dR_t = \frac{dS_t}{S_t} = \boldsymbol{\mu_t}\, dt + \sigma\, dW_t$	noisy observations

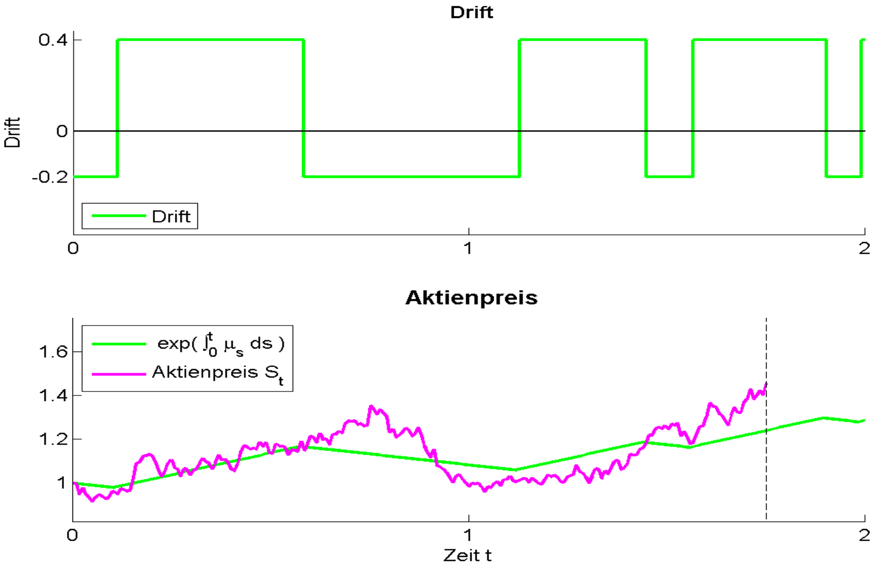
Given observations of prices S_u (returns R_u) for $u \in [0, t]$

Find filter for drift μ_t: $\widehat{\mu}_t = E\left[\mu_t | \mathcal{F}_t^S\right]$

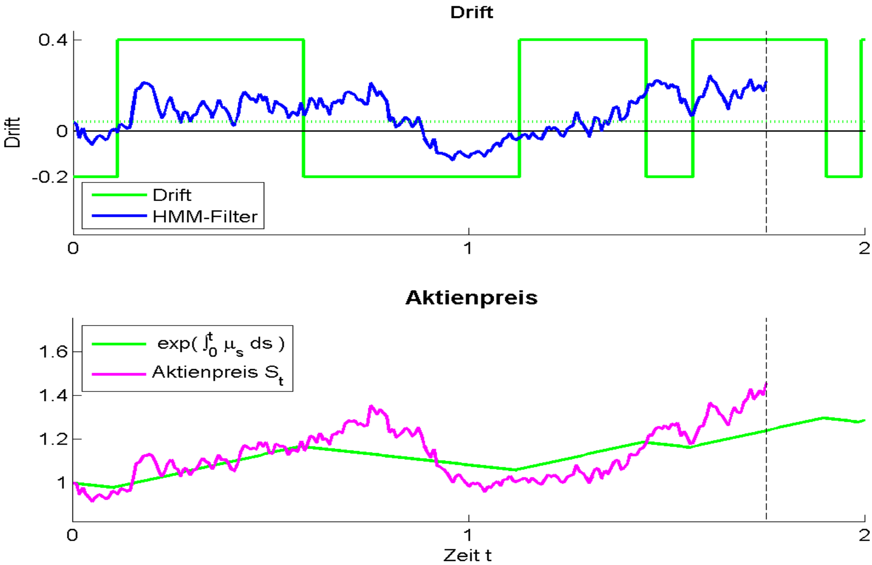
and martingale density Z_t: $\widehat{Z}_t = E\left[Z_t | \mathcal{F}_t^S\right]$

Solution Liptser, Shiryaev (1974), Wonham (1965), Elliot (1993)

HMM–Filter: Example



HMM–Filter: Example



Computation of the Optimal Strategy

Transformation to a model with full information

$$(\Omega, \mathbb{F}, P;\ \mathbf{Z}) \ \underset{\text{filter}}{\longrightarrow}\ (\Omega, \mathbb{F}^S, P|_{\mathbb{F}^S_T};\ \widehat{\mathbf{Z}})$$

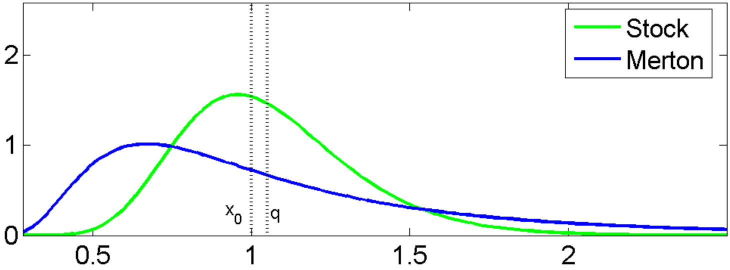
Optimal terminal wealth $\quad X_T^* \; = \; f(\widehat{\mathbf{Z}}_{\mathbf{T}}; y_1^*, y_2^*)$

Optimal strategy $\quad \pi_t^* \; = \; \sigma^{-\tau}\, \widetilde{E}\left[f'(\widehat{\mathbf{Z}}_{\mathbf{T}})\, D_t \widehat{\mathbf{Z}}_{\mathbf{T}} \,|\, \mathcal{F}_t^S\right] \qquad (*)$

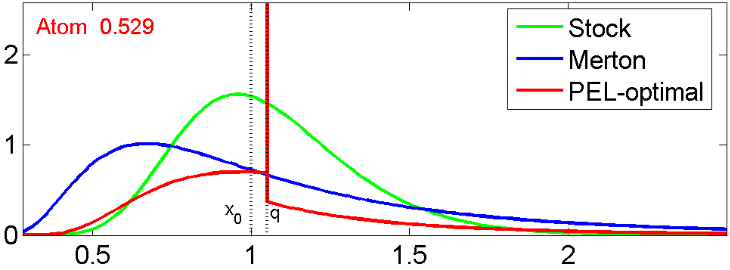
Computation of strategy requires

- Numerical solution of SDE's for the Malliavin Derivative $D_t \widehat{Z}_T$ using robust filter techniques
- Approximation of the conditional expectation in $(*)$ using Monte-Carlo simulation
- see Sass, Wunderlich (2010)

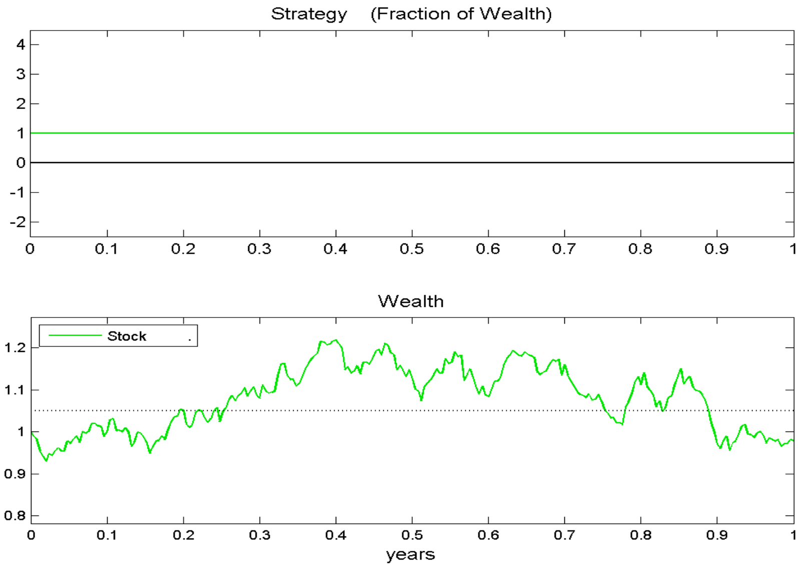
Distribution of Terminal Wealth



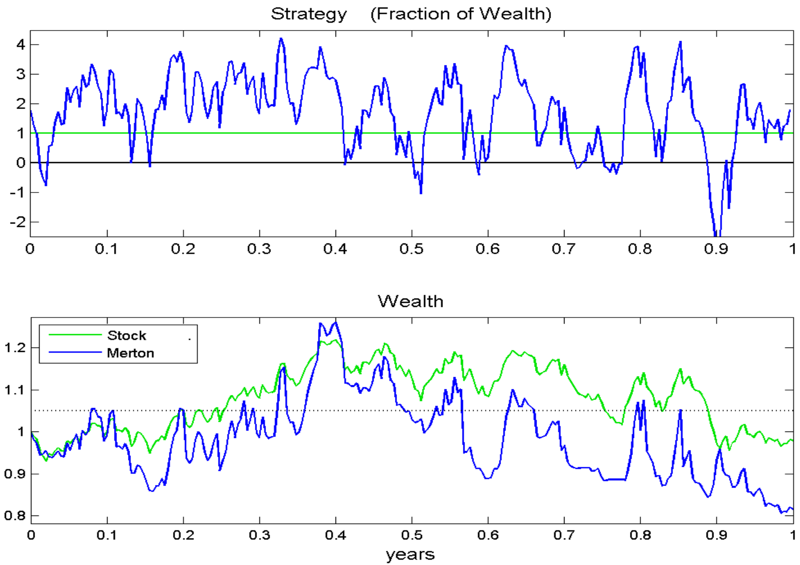
Distribution of Terminal Wealth



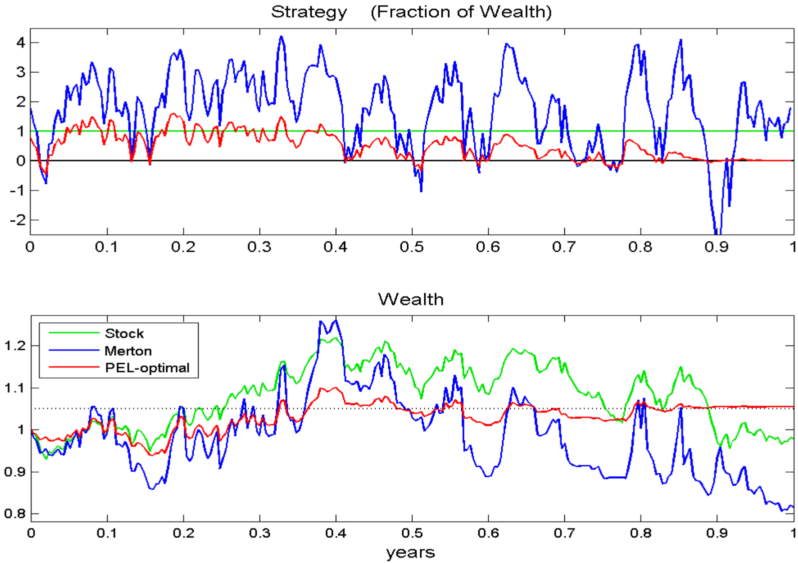
Optimal Strategy



Optimal Strategy



Optimal Strategy



Conclusion

- Dynamic portfolio optimization under risk constraint

$$\max_{\pi \in \mathcal{A}(x_0)} \quad E\left[U(X_T^\pi)\right], \qquad E_Q\left[(X_T^\pi - q)^-\right] \leq \varepsilon$$

- Partial information on the drift (Hidden Markov Model)

  X_T^* as a function of $\widehat{Z}_T$, the filter for the martingale density Z_T

  π_t^* depends on Malliavin derivative $D_t\widehat{Z}_T$

- Optimal strategies can be computed using Monte-Carlo simulations

References

- Basak, S. and Shapiro, A. (2001): Value-at-Risk-Based Risk Management: Optimal Policies and Asset Prices. *The Review of Financial Studies,* 2(14), 371-405, 2001.
- Gabih, A., Grecksch, W. and Wunderlich, R. (2005): *Dynamic portfolio optimization with bounded shortfall risks. Stochastic Analysis and Applications* 23, 579–594.
- Gabih, A., Sass, J. and Wunderlich, R. (2009): Utility maximization under bounded expected loss. *Stochastic Models* , 3(25), 2009, 375 - 407.
- Frey, R., Gabih, A. and Wunderlich, R. (2012): Portfolio Optimization under Partial Information with Expert Opinions. *International Journal of Theoretical and Applied Finance*,Vol. 15, No. 1.
- Sass, J. and Haussmann, U.G (2004): Optimizing the terminal wealth under partial information: The drift process as a continuous time Markov chain. *Finance and Stochastics* 8, 553–577.
- Sass, J. and Wunderlich, R. (2010): Optimal Portfolio Policies Under Bounded Expected Loss and Partial Information. *Mathematical Methods of Operations Research*, 72, 25–61