Optimal Portfolio Choice for Pension Funds

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Outline

• Why optimal portfolio choice?
• Classical Merton solution
• Static Formulation of Portfolio Choice
• Examples and Applications
• Conclusion
WHY OPTIMAL PORTFOLIO CHOICE?
Why Optimal Portfolio Choice?

• Important class of problems for pension funds, insurance companies and banks
  - Maximise expected return on investments
    - Subject to value-at-risk (or other) constraints
  - Asset-Liability Management
  - Life-cycle consumption and investment
  - Invest towards a benchmark at retirement
Optimal Portfolio Choice for Pension Funds

• Dutch pension discussion (super-brief summary)
• System with nominal guarantees
  - This is like a “nominal DB” system, with uncertain indexation
  - Extremely low interest rates make guarantees expensive
• “New pension deal” in early 2019
  - More DC system, remove nominal guarantees from system
  - This leads to shifting risk towards participants

• Our contribution:
  - Consider optimal investing towards a “DB target” within a DC setting
General Problem Formulation

• Specify a “utility function” $U(t, x)$
  
  $\max_{\{c_t\}, X_T} \mathbb{E} \left[ \int_0^T U(t, c_t) dt + U(T, X_T) \right]$

• Maximise life-time utility of consumption-process $\{c_t\}$ and terminal wealth $X_T$
  
  - Utility functions must be concave for all $t, x$
  - $U''(x) > 0$ and $U'''(x) < 0$ for all $x$ (and all $t$)
Some Examples

• Maximise expected return of $X_T$:
  - $U(t,c)=0$ for $t<T$ and $U(T, X_T) = \ln(X_T)$
  - “Growth optimal portfolio”, “Kelly criterion”

• Maximise power-utility (CRRA) of consumption
  - $U(t,c) = e^{-\delta t} c^{(1-\gamma)}$
  - Constant Relative Risk Aversion $\gamma$

• Minimise underfunding w.r.t. (random) target $Y_T$
  - $U(T,X_T) = \min( X_T - Y_T, 0)$
Terminal Wealth

• For this presentation, we focus on the “terminal wealth” case only
  - Focus on the essentials
  - Consumption solution has similar structure

• Hence: we focus on problems like:
  - $\max_{X_T} \mathbb{E}[U(X_T)]$
CLASSICAL MERTON SOLUTION
Robert Merton

- Problem first formulated and solved by Robert Merton in 1969
  - Book “Continuous-Time Finance”
  - Nobel prize Economics 1997 with Myron Scholes for Black-Scholes option price formula
  - Founding partner of Long-Term Capital Management (LTCM)
Black-Scholes Economy

• We assume an economy with 2 traded assets
  - Risk-free bank account $B_t$ with risk-free rate $r$
    - Value equation: $dB_t = rB_t \, dt$ or $B(t) = e^{rt}$
  - Stock $S_t$ (e.g. stock-market index)
    - Value equation: $dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$
  - Can generalise to multiple asset(-classes)

• BS economy has constant parameters: $r, \mu, \sigma$
Wealth Equation

- We are looking for an optimal investment strategy.
- Start with initial wealth $X_0$.
- Invest each time $t$ an amount $\pi_t$ in stocks.
  - Invest remainder $X_t - \pi_t$ in bank-account.
- Wealth equation:
  - $dX_t = (rX_t + (\mu - r)\pi_t)dt + \sigma\pi_t \, dW_t$
Merton Portfolio Problem

• Formulate the investment problem as a stochastic optimal control problem:
  \[
  \max \{ \pi_t \} \mathbb{E}[U(X_T)] - s.t. \quad dX_t = (rX_t + (\mu - r)\pi_t)dt + \sigma\pi_t \, dW_t
  \]

• Maximise expected utility of terminal wealth \( X_T \)
  - Using \( \pi_t \) as the control variable
  - Larger \( \pi_t \): higher return, but also more risk
Value Function

• Solve stochastic optimal control problem via backward induction:
  - Define value function $V(t, x) := \mathbb{E}_t [U(X_T) | X_t = x]$
  - Compute optimal value at time $t$ and wealth $x$, assuming that we follow optimal investment $\{\pi_s\}$ for all $s > t$
    - Bellman’s principle of optimality
• Derive pde for value function (Feynman-Kaç formula):
  - $V_t + (rx + (\mu - r)\pi_t)V_x + \frac{1}{2}\sigma^2\pi_t^2V_{xx} = 0$
  - Subscripts on $V$ denote partial derivatives w.r.t. $t$ and $x$
HJB equation

• Maximise the value-function using the Hamilton-Jacobi-Bellman equation:
  
  \[ V_t + \max_{\pi_t} \left\{ (rx + (\mu - r)\pi_t)V_x + \frac{1}{2}\sigma^2\pi_t^2 V_{xx} \right\} = 0 \]
  
  - Note: \( V_x(t, x) > 0 \) and \( V_{xx}(t, x) < 0 \) for all \( t, x \)
  - Choose optimal \( \pi_t \) for each time \( t \)

  \[ \pi^*(t, x) = \left( \frac{\mu-r}{\sigma^2} \right) \frac{V_x(t, x)}{-V_{xx}(t, x)} \quad (V\text{-ratio positive for all } t, x) \]

• This is the “easy part” for any control problem
HJB equation (2)

- The optimised value-function $V^*(t,x)$ follows a **non-linear** pde
  - This is the HJB equation for the Merton problem
    - $V_t + r x V_x - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 \frac{V_x^2}{V_{xx}} = 0$
    - non-linear term: $\frac{V_x^2(t,x)}{V_{xx}(t,x)}$

- Non-linear pde’s are hard to solve 😞

- This is the “hard part” of HJB
  - Most HJB equations cannot be solved analytically
Merton’s Solution

• However, Merton (1969) solved the problem analytically for power-utility → Major result! 😊
• For power-util we have: $U(X) = X^{(1-\gamma)}$

“Guess” the functional form: $V(t, x) = h(t)x^{(1-\gamma)}$ with $h(T)=1$
  - $V_t = \dot{h}(t)x^{1-\gamma}, V_x = h(t)(1-\gamma)x^{-\gamma}, V_{xx} = h(t)(1-\gamma)(-\gamma)x^{-\gamma-1}$

• For this guess we find for the non-linear term:
  - $\frac{V_x^2}{V_{xx}} = \frac{h(t)^2(1-\gamma)^2x^{-2\gamma}}{h(t)(1-\gamma)(-\gamma)x^{-\gamma-1}} = \frac{\gamma-1}{\gamma} h(t)x^{(1-\gamma)} = \frac{\gamma-1}{\gamma} V(t, x)$
Merton’s Solution (2)

- The non-linear HJB equation reduces to
  \[
  \dot{h}(t) + \left( r(1 - \gamma) - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 \left( \frac{\gamma - 1}{\gamma} \right) \right) h(t) = 0 \quad \text{ode in } h(t)
  \]

- Solution: \[ h(t) = e^{-(\gamma - 1) \left( r + \frac{1}{2} \frac{(\mu - r)^2}{\gamma \sigma^2} \right)(T - t)} \]

- Solution for optimal value-function:
  \[
  V^*(t, x) = e^{-(\gamma - 1) \left( r + \frac{1}{2} \frac{(\mu - r)^2}{\gamma \sigma^2} \right)(T - t)} x(1 - \gamma)
  \]
Merton’s Solution (3)

• Optimal investment policy:
  \[ \pi^*(t, x) = \left( \frac{\mu - r}{\sigma^2} \right) \frac{V_x^*(t,x)}{-V_{xx}^*(t,x)} = \left( \frac{\mu - r}{\gamma \sigma^2} \right) x \]

• Remarkably beautiful and simple result:
  - Always invest fixed proportion of wealth \( x \) in stocks
  - Increase in \( (\mu - r) \): excess return of stocks
  - Decrease in \( \gamma \sigma^2 \): risk-aversion, volatility of stocks

• This is basis for “fix-mix” investment strategies
Merton Solution Summary

• Beautiful and remarkable result
  - “Fix-mix” investment strategy is optimal
• But...
  - We must “guess” a $V^*(t,x)$ to solve the HJB equation
  - Need $V^*(t,x)$ to find optimal investment strategy
• Difficult to solve more complicated versions
  - Different utility functions
  - Non-constant $r, \mu, \sigma$
STATIC FORMULATION OF PORTFOLIO CHOICE
Chris Rogers

• Interesting monograph by Chris Rogers
• Different solution approaches:
  - HJB, Static, Duality
• Consider 34 variations of optimal investment
A Different Perspective

• With the HJB equation, we make a “detour” via the value-function to obtain optimal policy
  - But, we are really interested in the optimal policy, and not in the value function

• New perspective in late 1980’s
  - Cox-Huang, J Econ Theory (1989)
  - “Martingale formulation” or “Static formulation”
Terminal Wealth Problem

• Consider the stochastic optimal control problem:
  - \[ \max_{\{\pi_t\}} \mathbb{E}[U(X_T)] \]
  - s.t. \( X_T = X_0 + \int_0^T (rX_t + (\mu - r)\pi_t)dt + \int_0^T \sigma\pi_t \, dW_t \)

• Note, we have expressed the wealth explicitly as a stochastic integral
  - Via \( \{\pi_t\} \) we control the terminal wealth \( X_T \)
Change of Variables

• Simplify budget constraint by considering
  - $\bar{X}_T := \frac{X_T}{B_T} \Rightarrow$ use bank-account $B_t$ as numéraire
  - Ito’s Lemma: $\bar{X}_T = X_0 + \int_0^T (\mu - r)\bar{\pi}_t \, dt + \int_0^T \sigma\bar{\pi}_t \, dW_t$
  - Note: also valid for stochastic $r_t$

• Rewrite stochastic integral as:
  - $\bar{X}_T = X_0 + \int_0^T \bar{\pi}_t (\sigma dW_t + (\mu - r)\, dt)$
  - Integrate over stochastic returns $(\sigma dW_t + (\mu - r)\, dt)$
Lagrange Formulation

• New perspective: consider wealth equation as a linear constraint on terminal wealth $X_T$
• We then obtain an optimisation problem with an equality constraint
  
  $$\max_{\bar{X}_T, \{\bar{\pi}_t\}} \mathbb{E}[U(B_T \bar{X}_T)]$$

  - s.t.  
  $$\bar{X}_T = X_0 + \int_0^T \bar{\pi}_t (\sigma dW_t + (\mu - r) dt)$$

- Decision variables: $\bar{X}_T$ and $\{\bar{\pi}_t\}_{0 \leq t \leq T}$ with linear constraint
- Solve with Lagrange’s method
Lagrange Formulation (2)

- **Intuition:** Consider collection of \( n=1..N \) paths for the asset-returns
  
  \[
  \max_{\tilde{\pi}_{T,n}\{\pi_{t,n}\}} \sum_{n=1}^{N} \frac{1}{N} U(B_{T,n}\tilde{X}_{T,n})
  \]
  
  \[
  \text{s.t. } \tilde{X}_{T,n} = X_0 + \int_0^T \tilde{\pi}_{t,n}(\sigma dW_{t,n} + (\mu - r) dt) \quad \forall \ n = 1..N
  \]

- **Wealth-equation has to hold for all paths \( n = 1..N \)**
  
  - Collection of \( N \) equality-constraints

- **Introduce \( N \) Lagrange multipliers \( \Lambda_n \) to build:**
  
  \[
  \mathcal{L}\{\{\tilde{\pi}_{t,n}\},\tilde{X}_{T,n},\Lambda_{T,n}\} := \sum_{n=1}^{N} \frac{1}{N} U(B_{T,n}\tilde{X}_{T,n}) - \Lambda_n (\tilde{X}_{T,n} - X_0 - \int_0^T \tilde{\pi}_{t,n}(\sigma dW_{t,n} + (\mu - r) dt))
  \]
Lagrange Formulation (3)

• **Continuum:** wealth-equation has to hold for all states of the world \( \omega \in \Omega \)

• Introduce collection of Lagrange multipliers \( \Lambda_T(\omega) \)
  - This is a random variable, measurable w.r.t. \( \mathcal{F}_T \)

• Lagrange function:
  - \( \mathcal{L}(\{\pi_t\}, \bar{X}_T, \Lambda_T) := \mathbb{E} \left[ U(B_T \bar{X}_T) - \Lambda_T \left( \bar{X}_T - X_0 - \int_0^T \pi_t (\sigma dW_t + (\mu - r) dt) \right) \right] \)
  - “\( \mathbb{E}[\Lambda_T (\cdots)] \)” performs the summation over all \( \omega \in \Omega \)
Lagrange Solution

• We can now maximise the Lagrange-function
  \[ 
  \mathcal{L}(\{\pi_t\}, \bar{X}_T, \Lambda_T) := \mathbb{E} \left[ U(B_T \bar{X}_T) - \Lambda_T \left( \bar{X}_T - X_0 - \int_0^T \sigma \pi_t (dW_t + \frac{\mu - r}{\sigma} dt) \right) \right] 
  \]
  - Unconstrained optimisation problem in
    \[ \{\pi_t\}, \bar{X}_T, \Lambda_T \]
  - Lagrange-function \( \mathcal{L}(\cdot) \) is linear in \( \pi_t \)
  - Obtain finite value for \( \mathcal{L}(\cdot) \) only when
    \[ 
    \mathbb{E} \left[ \Lambda_T \int_0^T \sigma \pi_t (dW_t + \frac{\mu - r}{\sigma} dt) \right] = 0 \text{ for all } \pi_t \]
Choice for $Λ_T$

- We want $\mathbb{E} \left[ Λ_T \int_0^T σ \bar{π}_t \left( dW_t + \frac{µ-r}{σ} dt \right) \right] = 0$ for all $\bar{π}_t$
  - Assume $Λ_T > 0$, then $\frac{Λ_T}{\mathbb{E}[Λ_T]}$ is a valid Radon-Nikodym derivative that defines a new probability measure
- Select the measure $\mathbb{Q}$ with $dW_t + \frac{µ-r}{σ} dt \rightarrow dW_t^\mathbb{Q}$
  - Extension of this result for incomplete market possible
- Then, integrator $dW_t^\mathbb{Q}$ is a $\mathbb{Q}$-martingale
  - Measure $\mathbb{Q}$ is “the” risk-neutral measure!
  - For Black-Scholes: $\mathbb{Q}_T = Ce^{-\left(\frac{µ-r}{σ}\right)W_T}$ is a lognormal r.v.
“Reduced” Lagrange Form

• When we choose $\Lambda_T = \Lambda_0 Q_T$ we obtain
  
  $$
  \mathbb{E} \left[ \Lambda_T \int_0^T \sigma \tilde{\pi}_t \left( dW_t + \frac{\mu - \gamma}{\sigma} dt \right) \right] = \Lambda_0 \mathbb{E}^Q \left[ \int_0^T \sigma \tilde{\pi}_t \, dW_t^Q \right] = 0 \text{ for all } \tilde{\pi}_t
  $$

• We now know $\Lambda_T$ up to scaling constant $\Lambda_0$

• Consider “reduced” Lagrange-function:
  
  $$
  \tilde{\mathcal{L}}(\bar{X}_T, \Lambda_0) := \mathbb{E} \left[ U(B_T \bar{X}_T) - \Lambda_0 Q_T (\bar{X}_T - X_0) \right]
  $$

• Rewrite as:
  
  $$
  \tilde{\mathcal{L}}(X_T, \Lambda_0) = \mathbb{E} \left[ U(X_T) \right] - \Lambda_0 \left( \mathbb{E}^Q \left[ \frac{X_T}{B_T} \right] - X_0 \right)
  $$
Martingale Formulation

• Formulate primal problem in “martingale form”:
  - \[ \max_{X_T} \mathbb{E}[U(X_T)] \quad \text{s.t.} \quad \mathbb{E}^\mathbb{Q}\left[\frac{X_T}{B_T}\right] = X_0 \]
  - Valid formulation for complete market (i.e. unique \( \mathbb{Q} \))

• Maximise \( \tilde{\mathcal{L}}() \) for \( X_T \): 
  - \( U'(X_T(\omega)) - \Lambda_0 \frac{\mathbb{Q}_T(\omega)}{B_T(\omega)} = 0 \)
  - Intuition: increase util, but at a “\( \mathbb{Q} \)-price” in state \( \omega \)
  - Solution: \( X_T^* = I\left(\Lambda_0 \frac{\mathbb{Q}_T}{B_T}\right) \) \( I() \) is inverse function of \( U'(\) \)
  - Solve scalar \( \Lambda_0 \) such that \( X_T^* \) satisfies budget constraint
Martingale Formulation (2)

• The solution: \( X_T^* = I \left( \Lambda_0 \frac{Q_T}{B_T} \right) \) is extremely general
  - Holds for any (strictly concave) utility function \( U() \)
  - Holds for any pricing kernel \( Q_T / B_T \)
    - Even with stochastic interest rates, stochastic volatility, etc
  - Can extend this method to incomplete markets: Kamma & Pelsser (2019)

• However...
  - Find scalar \( \Lambda_0 \) such that \( X_T^* \) satisfies budget constraint
  - Must do this numerically for most models
Merton Portfolio Problem

- For the Black-Scholes economy we have \( \mathbb{Q}_T \propto \exp\left\{ -\left(\frac{\mu-r}{\sigma}\right)W_T \right\} \)
- Power util: \( U(x) = \frac{x^{(1-\gamma)-1}}{(1-\gamma)} \) with \( U'(x) = x^{-\gamma} \) and \( I(y) = y^{-\frac{1}{\gamma}} \)

- Optimal wealth: \( X_T^* = I \left( \Lambda_0 \frac{\mathbb{Q}_T}{B_T} \right) = Ce^{\frac{\mu-r}{\gamma \sigma}W_T} = \tilde{C} \left( S_T \right)^{\frac{\mu-r}{\gamma \sigma^2}} \)
  - Delta-hedge \( X_T^* \) by holding \( \Delta_t = \frac{\partial X_t^*}{\partial S_t} = \left( \frac{\mu-r}{\gamma \sigma^2} \right) \frac{X_t^*}{S_t} \) units of \( S_t \)
  - Replicate \( X_T^* \) by investing portion \( \frac{\mu-r}{\gamma \sigma^2} \) of wealth in stocks
Minimise Underfunding

- Minimise underfunding w.r.t. (random) target \( Y_T \)
  - \( U_{ES}(X_T) = \min(X_T - Y_T, 0) \)

- Consider \( \mathbb{E}^Q \left[ \frac{Y_T}{B_T} \right] > X_0 \) : \( Y_T \) is more expensive than \( X_0 \)
  - \( U'(X_T) = \mathbb{1}_{X_T < Y_T} \) with inverse function
  - \( I(y) = Y_T \) for \( y \leq 1 \) and \( I(y) = 0 \) for \( y > 1 \)

- \( X_T^* = I \left( \Lambda_0 \frac{Q_T}{B_T} \right) = Y_T \mathbb{1} \left( \frac{Q_T}{B_T} \leq C \right), \) solve \( C \) for budget \( X_0 \)
  - Replicate \( Y_T \) except for “expensive” states: \( \frac{Q_T}{B_T} > C \)
Minimise Underfunding (2)

• Optimal payoff, that minimises expected shortfall
• With only 75% budget
• See: Föllmer-Leukert (2000)
Minimise Underfunding of DB

• Black-Scholes-Vasicek model
  - Stochastic stock return and stochastic (real) interest rates
• Target $Y_T$ is a (real) annuity at retirement age $T$
  - Level of annuity depends on (real) interest rate
    - And mortality... which we ignore for now

• Optimal investment towards a Defined Benefit target $Y_T$
  within a Defined Contribution budget
Minimise Underfunding DB (2)

- Assume the NPV of all premiums only finances 50% of annuity market-value $\mathbb{E}^Q[e^{-rT}Y_T]$ at $t=0$
  - Find optimal investment $X_T$ that minimises expected shortfall: $\min(X_T - Y_T, 0)$
  - Horizon of $T=40$ years
- Optimal investment strategy: achieves “success ratio” of over 95% (!)
Vector-AR model

- Many ALM models have multiple state-variables:
  - Nominal interest rates, stocks, real-estate, inflation, etc
- Suppose we have a state-vector $Y_t$ that evolves as VAR model (or vector-OU model):
  - $dY_t = (\theta - AY_t)dt + \Sigma \cdot dW_t^P$
- Change of measure: $dQ_t = Q_t \kappa' dW_t$ then:
  - $dY_t = ((\theta + \Sigma \kappa) - AY_t)dt + \Sigma \cdot dW_t^Q$
  - For constant $\kappa$, $Q_t$ is a lognormal process
VAR model (2)

• We can solve the optimal investment for general utility $U()$:
  - Optimal wealth: $X_T^* = I \left( \Lambda_0 \frac{Q_T}{B_T} \right)$
  - Function $I()$ is inverse of marginal utility $U'(())$
  - Solve constant $\Lambda_0$ to fit budget constraint

• Investment strategy is given by “delta-hedging” current wealth: $X_t^* = \mathbb{E}_t^Q \left[ \frac{B_t}{B_T} I \left( \Lambda_0 \frac{Q_T}{B_T} \right) \right]$
  - Compute a “delta” w.r.t. each of the traded assets
VAR model (3)

- The model for the “Haalbaarheidstoets” (HBT) is a VAR-model
  - Prescribed by Dutch Central Bank (DNB) for pension funds
  - Stochastic model for stocks, interest rates, inflation
  - Model based on Koijen-Nijman-Werker, RFS (2010)

\[
\begin{align*}
    d & \begin{bmatrix} X \\ \ln \Pi \\ \ln S \\ \ln P^{F^0} \\ \ln P^{F^\tau} \end{bmatrix} = \\
    & \begin{bmatrix}
        0 & \delta_{0\pi} - \frac{1}{2} \sigma_{\Pi}^2 \sigma_{\Pi} & R_0 + \eta_S - \frac{1}{2} \sigma_S^2 \sigma_S \\
        \delta_{1\pi} & 0 & R_1 \\
        R_0 & 0 & \Sigma_X' \Lambda_0 - \frac{1}{2} B^N \Sigma_X \Sigma_X B^N \\
        R_1 & 0 & \Sigma_X' \Lambda_1 \\
        R_1 + B^N (\tau)' \Sigma_X \Lambda_1 & 0 & \Sigma_X' \\
    \end{bmatrix} \\
    & \begin{bmatrix} X \\ \ln \Pi \\ \ln S \\ \ln P^{F^0} \\ \ln P^{F^\tau} \end{bmatrix} + \\
    & \begin{bmatrix}
        -K & 0 \\
        0 & \Sigma_X' \\
        0 & \sigma_{\Pi} \\
        0 & \sigma_S' \\
        0 & 0 \\
    \end{bmatrix} \\
    & \begin{bmatrix} dT \\ dZ_t \end{bmatrix}
\end{align*}
\]
Conclusion

• With “martingale formulation” we are able to find general solution for optimal investment problem:
  - Holds for any (strictly concave) utility function $U()$
  - Holds for any pricing kernel $\mathbb{Q}_T/B_T$
    - Even with stochastic interest rates, stochastic volatility, etc

• VAR-models are explicitly solvable
  - Even for multiple assets, e.g. HBT model


Merton, R. C. (1990), Continuous-Time Finance, Blackwell
