Optimal Portfolio Choice for Pension Funds

prof.dr. Antoon Pelsser HonFIA

Pelsser Consulting
Maastricht University
Netspar

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Maastricht University

Outline

- Why optimal portfolio choice?
- Classical Merton solution
- Static Formulation of Portfolio Choice
- Examples and Applications
- Conclusion

WHY OPTIMAL PORTFOLIO CHOICE?

Why Optimal Portfolio Choice?

- Important class of problems for pension funds, insurance companies and banks
  - Maximise expected return on investments
    - Subject to value-at-risk (or other) constraints
  - Asset-Liability Management
  - Life-cycle consumption and investment
  - Invest towards a benchmark at retirement

Optimal Portfolio Choice for Pension Funds

- Dutch pension discussion (super-brief summary)
- System with nominal guarantees
  - This is like a "nominal DB" system, with uncertain indexation
  - Extremely low interest rates make guarantees expensive
- "New pension deal" in early 2019
  - More DC system, remove nominal guarantees from system
  - This leads to shifting risk towards participants

- Our contribution:
  - Consider optimal investing towards a "DB target" within a DC setting

General Problem Formulation

- Specify a "utility function" *U(t,x)*
  - $$\max_{\{c_t\},X_T} \mathbb{E}\left[\int_0^T U(t,c_t)dt \; + U(T,X_T)\right]$$
- Maximise life-time utility of consumption-process $\{c_t\}$ and terminal wealth X_T
  - Utility functions must be concave for all *t,x*
  - *U'(x)*>0 and *U''(x)*<0 for all *x* (and all *t*)

Some Examples

- Maximise expected return of X_T:
  - $U(t,c)=0$ $t<T$ and $U(T, X_T) = \ln(X_T)$
  - "Growth optimal portfolio", "Kelly criterion"
- Maximise power-utility (CRRA) of consumption
  - $U(t,c) = e^{-\delta t}\, c^{(1-\gamma)}$
  - Constant Relative Risk Aversion γ
- Minimise underfunding w.r.t. (random) target Y_T
  - $U(T,X_T) = \min(X_T - Y_T\, , 0)$

Terminal Wealth

- For this presentation, we focus on the "terminal wealth" case only
  - Focus on the essentials
  - Consumption solution has similar structure

- Hence: we focus on problems like:
  - $\max_{X_T} \mathbb{E}[U(X_T)]$

CLASSICAL MERTON SOLUTION

Robert Merton

- Problem first formulated and solved by Robert Merton in 1969
  - Book “Continuous-Time Finance”
  - Nobel prize Economics 1997 with Myron Scholes for Black-Scholes option price formula
  - Founding partner of Long-Term Capital Management (LTCM)

Black-Scholes Economy

- We assume an economy with 2 traded assets
  - Risk-free bank account B_t with risk-free rate r
    - Value equation: $dB_t = rB_t\, dt$ or $B(t) = e^{rt}$
  - Stock S_t (e.g. stock-market index)
    - Value equation: $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$
  - Can generalise to multiple asset(-classes)
- BS economy has constant parameters: r, μ, σ

Wealth Equation

- We a looking for an optimal investment strategy
- Start with initial wealth X_0.
- Invest each time t an amount π_t in stocks
  - Invest remainder $X_t - \pi_t$ in bank-account
- Wealth equation:
  - $dX_t = (rX_t + (\mu - r)\pi_t)dt + \sigma\pi_t\, dW_t$

Excess return

Volatility

Merton Portfolio Problem

- Formulate the investment problem as a stochastic optimal control problem:
  - $\max_{\{\pi_t\}} \mathbb{E}[U(X_T)]$
  - $s.t.\ dX_t = (rX_t + (\mu - r)\pi_t)dt + \sigma\pi_t\, dW_t$

- Maximise expected utility of terminal wealth X_T
  - Using π_t as the control variable
  - Larger π_t: higher return, but also more risk

Value Function

- Solve stochastic optimal control problem via backward induction:
  - Define value function $V(t,x) \coloneqq \mathbb{E}_t[U(X_T) \mid X_t = x]$
  - Compute optimal value at time *t* and wealth *x*, assuming that we follow optimal investment $\{\pi_s\}$ for all *s>t*
    - Bellman's principle of optimality
- Derive pde for value function (Feynman-Kaç formula):
  - $V_t + (rx + (\mu - r\ )\pi_t)V_x + \frac{1}{2}\sigma^2\pi_t^2 V_{xx} = 0$
  - Subscripts on *V* denote partial derivatives w.r.t. *t* and *x*

HJB equation

- Maximise the value-function using the Hamilton-Jacobi-Bellman equation:
  - $V_t + \max_{\pi_t} \left\{ (rx + (\mu - r)\pi_t)V_x + \frac{1}{2}\sigma^2 \pi_t^2 V_{xx} \right\} = 0$
  - Note: $V_x(t,x) > 0$ and $V_{xx}(t,x) < 0$ for all *t*,*x*
  - Choose optimal π_t for each time *t*
  - $\pi^*(t,x) = \left(\frac{\mu - r}{\sigma^2}\right) \frac{V_x(t,x)}{-V_{xx}(t,x)}$ (*V*-ratio positive for all *t*,*x*)

- This is the “easy part” for any control problem

HJB equation (2)

- The optimised value-function $V^*(t,x)$ follows a <u>non-linear</u> pde
  - This is the HJB equation for the Merton problem
  - $V_t + rxV_x - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{V_x^2}{V_{xx}} = 0$ non-linear term: $\frac{V_x^2(t,x)}{V_{xx}(t,x)}$
- Non-linear pde's are hard to solve ☹

- This is the "hard part" of HJB
  - Most HJB equations cannot be solved analytically

Merton's Solution

- However, Merton (1969) solved the problem analytically for power-utility → Major result! ☺
- For power-util we have: $U(X) = X^{(1-\gamma)}$

- "Guess" the functional form: $V(t,x) = h(t)x^{(1-\gamma)}$ with *h(T)*=1
  - $V_t = \dot{h}(t)\, x^{1-\gamma}, V_x = h(t)\,(1-\gamma)x^{-\gamma}, V_{xx} = h(t)\,(1-\gamma)(-\gamma)x^{-\gamma-1}$

- For this guess we find for the non-linear term:
  - $\frac{V_x^2}{V_{xx}} = \frac{h(t)^2(1-\gamma)^2 x^{-2\gamma}}{h(t)\,(1-\gamma)(-\gamma)x^{-\gamma-1}} = \frac{\gamma-1}{\gamma} h(t)x^{(1-\gamma)} = \frac{\gamma-1}{\gamma} V(t,x)$

Merton's Solution (2)

- The non-linear HJB equation reduces to
  - $\dot{h}(t) + \left(r(1-\gamma) - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 \left(\frac{\gamma-1}{\gamma}\right) \right) h(t) = 0$ ode in *h(t)*
  - Solution: $h(t) = e^{-(\gamma-1)\left(r+\frac{1}{2}\frac{(\mu-r)^2}{\gamma\sigma^2}\right)(T-t)}$
- Solution for optimal value-function:
  - $V^*(t,x) = e^{-(\gamma-1)\left(r+\frac{1}{2}\frac{(\mu-r)^2}{\gamma\sigma^2}\right)(T-t)} x^{(1-\gamma)}$

Merton's Solution (3)

- Optimal investment policy:
  - $\pi^*(t,x) = \left(\frac{\mu - r}{\sigma^2}\right) \frac{V_x^*(t,x)}{-V_{xx}^*(t,x)} = \left(\frac{\mu - r}{\gamma\sigma^2}\right) x$
- Remarkably beautiful and simple result:
  - Always invest fixed proportion of wealth *x* in stocks
  - Increase in $(\mu - r)$: excess return of stocks
  - Decrease in $\gamma\sigma^2$: risk-aversion, volatility of stocks
- This is basis for "fix-mix" investment strategies

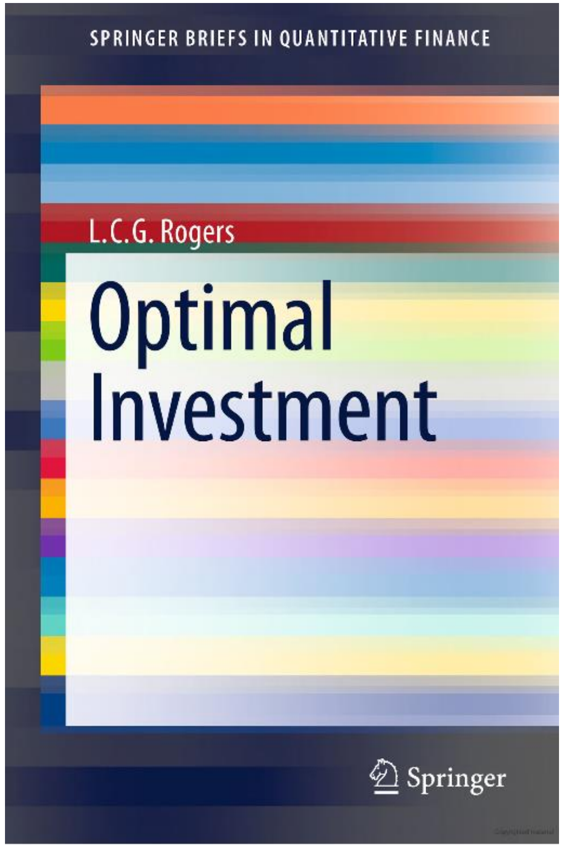
Merton Solution Summary

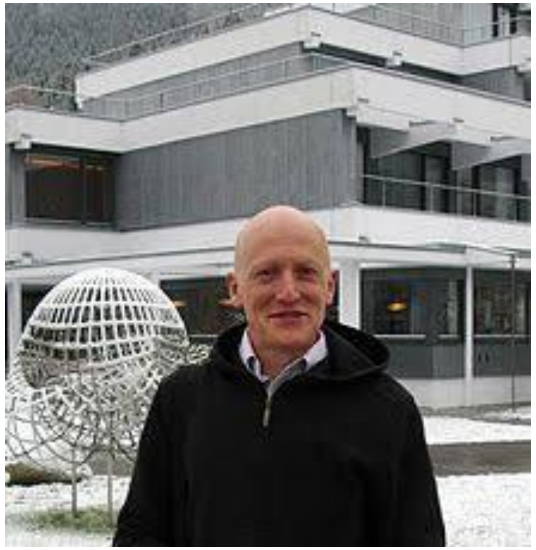
- Beautiful and remarkable result
  - "Fix-mix" investment strategy is optimal
- But…
  - We must "guess" a $V^*(t,x)$ to solve the HJB equation
  - Need $V^*(t,x)$ to find optimal investment strategy
- Difficult to solve more complicated versions
  - Different utility functions
  - Non-constant r, μ, σ

STATIC FORMULATION OF PORTFOLIO CHOICE

Chris Rogers

- Interesting monograph by Chris Rogers
- Different solution approaches:
  - HJB, Static, Duality
- Consider 34 variations of optimal investment





A Different Perspective

- With the HJB equation, we make a “detour” via the value-function to obtain optimal policy
  - But, we are really interested in the optimal policy, and not in the value function

- New perspective in late 1980’s
  - Pliska, *Math Operations Res* (1986)
  - Karatzas-Lehoczky-Shreve, *SIAM J Optim Cont* (1987)
  - Cox-Huang, *J Econon Theory* (1989)
  - “Martingale formulation” or “Static formulation”

Terminal Wealth Problem

- Consider the stochastic optimal control problem:
  - $\max_{\{\pi_t\}} \mathbb{E}[U(X_T)]$
  - $s.t.\ X_T = X_0 + \int_0^T (rX_t + (\mu - r)\pi_t)dt + \int_0^T \sigma\pi_t\, dW_t$

- Note, we have expressed the wealth explicitly as a stochastic integral
  - Via $\{\pi_t\}$ we control the terminal wealth X_T

Change of Variables

- Simplify budget constraint by considering
  - $\bar{X}_T := \frac{X_T}{B_T} \rightarrow$ use bank-account B_t as numéraire
  - Ito's Lemma: $\bar{X}_T = X_0 + \int_0^T (\mu - r)\bar{\pi}_t \, dt + \int_0^T \sigma \bar{\pi}_t \, dW_t$
  - Note: also valid for stochastic r_t

- Rewrite stochastic integral as:
  - $\bar{X}_T = X_0 + \int_0^T \bar{\pi}_t(\sigma dW_t + (\mu - r)dt)$
  - Integrate over stochastic returns $(\sigma dW_t + (\mu - r)dt)$

Lagrange Formulation

- New perspective: consider wealth equation as a linear constraint on terminal wealth XT
- We then obtain an optimisation problem with an equality constraint
  - $\max\limits_{\bar{X}_T, \{\bar{\pi}_t\}} \mathbb{E}[U(B_T \bar{X}_T)]$
  - $s.t. \quad \bar{X}_T = X_0 + \int_0^T \bar{\pi}_t(\sigma dW_t + (\mu - r)dt)$
  - Decision variables: $\bar{X}_T$ and $\{\bar{\pi}_t\}_{0 \le t \le T}$ with linear constraint
  - Solve with Lagrange's method

Lagrange Formulation (2)

- **Intuition**: Consider collection of *n=1..N* paths for the asset-returns
    - $\max_{\bar{X}_{T,n},\{\bar{\pi}_{t,n}\}} \sum_{n=1}^{N} \frac{1}{N} U(B_{T,n}\bar{X}_{T,n})$
    - $s.t. \quad \bar{X}_{T,n} = X_0 + \int_0^T \bar{\pi}_{t,n}\big(\sigma dW_{t,n} + (\mu - r)dt\big) \;\; \forall\, n = 1..N$
- Wealth-equation has to hold for all paths $n = 1..N$
    - Collection of *N* equality-constraints
- Introduce *N* Lagrange multipliers Λ_n to build:
    - $\mathcal{L}\big(\{\bar{\pi}_{t,n}\}, \bar{X}_{T,n}, \Lambda_{T,n}\big) \coloneqq \sum_{n=1}^{N} \frac{1}{N} U(B_{T,n}\bar{X}_{T,n}) - \Lambda_n\big(\bar{X}_{T,n} - X_0 - \int_0^T \bar{\pi}_{t,n}(\sigma dW_{t,n} + (\mu - r)dt)\big)$

Lagrange Formulation (3)

- **Continuum:** wealth-equation has to hold for all states of the world $\omega \in \Omega$
- Introduce collection of Lagrange multipliers $\Lambda_T(\omega)$
  - This is a random variable, measurable w.r.t. $\mathcal{F}_T$
- Lagrange function:
  - $\mathcal{L}(\{\bar{\pi}_t\}, \bar{X}_T, \Lambda_T) := \mathbb{E}\left[U(B_T\bar{X}_T) - \Lambda_T\left(\bar{X}_T - X_0 - \int_0^T \bar{\pi}_t(\sigma dW_t + (\mu - r)dt)\right)\right]$
  - "$\mathbb{E}[\Lambda_T\,(\cdots)]$" performs the summation over all $\omega \in \Omega$

Lagrange Solution

- We can now maximise the Lagrange-function
  - $\mathcal{L}(\{\bar{\pi}_t\}, \bar{X}_T, \Lambda_T) := \mathbb{E}\left[U(B_T \bar{X}_T) - \Lambda_T\left(\bar{X}_T - X_0 - \int_0^T \sigma\bar{\pi}_t\left(dW_t + \frac{\mu-r}{\sigma}dt\right)\right)\right]$
  - Unconstrained optimisation problem in $(\{\bar{\pi}_t\}, \bar{X}_T, \Lambda_T)$
- Lagrange-function $\mathcal{L}()$ is linear in $\bar{\pi}_t$
- Obtain finite value for $\mathcal{L}()$ only when
  - $\mathbb{E}\left[\Lambda_T \int_0^T \sigma\bar{\pi}_t\left(dW_t + \frac{\mu-r}{\sigma}dt\right)\right] = 0$ for all $\bar{\pi}_t$

Choice for Λ_T

- We want $\mathbb{E}\left[\Lambda_T \int_0^T \sigma\bar{\pi}_t\left(dW_t + \frac{\mu-r}{\sigma}dt\right)\right] = 0$ for all $\bar{\pi}_t$
  - Assume $\Lambda_T > 0$, then $\frac{\Lambda_T}{\mathbb{E}[\Lambda_T]}$ is a valid Radon-Nikodym derivative that defines a new probability measure
- Select the measure $\mathbb{Q}$ with $dW_t + \frac{\mu-r}{\sigma}dt \to dW_t^{\mathbb{Q}}$
  - Extension of this result for incomplete market possible
- Then, integrator $dW_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-martingale
  - Measure $\mathbb{Q}$ is “the” risk-neutral measure!
  - For Black-Scholes: $\mathbb{Q}_T = Ce^{-\left(\frac{\mu-r}{\sigma}\right)W_T}$ is a lognormal r.v.

"Reduced" Lagrange Form

- When we choose $\Lambda_T = \Lambda_0 \mathbb{Q}_T$ we obtain
  - $\mathbb{E}\left[\Lambda_T \int_0^T \sigma \bar{\pi}_t \left(dW_t + \frac{\mu - r}{\sigma} dt\right)\right] = \Lambda_0 \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \sigma \bar{\pi}_t \, dW_t^{\mathbb{Q}}\right] = 0$ for all $\bar{\pi}_t$
- We now know Λ_T up to scaling constant Λ_0
- Consider "reduced" Lagrange-function:
  - $\tilde{\mathcal{L}}(\bar{X}_T, \Lambda_0) := \mathbb{E}[U(B_T \bar{X}_T) - \Lambda_0 \mathbb{Q}_T(\bar{X}_T - X_0)]$
- Rewrite as:
  - $\tilde{\mathcal{L}}(X_T, \Lambda_0) = \mathbb{E}[U(X_T)] - \Lambda_0 \left(\mathbb{E}^{\mathbb{Q}}\left[\frac{X_T}{B_T}\right] - X_0\right)$

Martingale Formulation

- Formulate primal problem in "martingale form":
  - $\max_{X_T} \mathbb{E}[U(X_T)] \quad s.t. \, \mathbb{E}^{\mathbb{Q}}\left[\frac{X_T}{B_T}\right] = X_0$
  - Valid formulation for complete market (i.e. unique $\mathbb{Q}$)

- Maximise $\tilde{\mathcal{L}}()$ for X_T: $U'\left(X_T(\omega)\right) - \Lambda_0 \frac{\mathbb{Q}_T(\omega)}{B_T(\omega)} = 0$
  - Intuition: increase util, but at a "$\mathbb{Q}$-price" in state ω
  - Solution: $X_T^* = I\left(\Lambda_0 \frac{\mathbb{Q}_T}{B_T}\right)$ *I() is inverse function of U'()*
  - Solve scalar Λ_0 such that X_T^* satisfies budget constraint

Martingale Formulation (2)

- The solution: $X_T^* = I\left(\Lambda_0 \frac{\mathbb{Q}_T}{B_T}\right)$ is extremely general
  - Holds for any (strictly concave) utility function *U()*
  - Holds for any pricing kernel $\mathbb{Q}_T / B_T$
    - Even with stochastic interest rates, stochastic volatility, etc
  - Can extend this method to incomplete markets: Kamma & Pelsser (2019)

- However…
  - Find scalar Λ_0 such that X_T^* satisfies budget constraint
  - Must do this numerically for most models

EXAMPLES AND APPLICATIONS

Merton Portfolio Problem

- For the Black-Scholes economy we have
  - $\mathbb{Q}_T \propto \exp\{-(\frac{\mu-r}{\sigma})W_T\}$
  - Power util: $U(x) = \frac{x^{(1-\gamma)}-1}{(1-\gamma)}$ with $U'(x) = x^{-\gamma}$ and $I(y) = y^{-\frac{1}{\gamma}}$

- Optimal wealth: $X_T^* = I\left(\Lambda_0 \frac{\mathbb{Q}_T}{B_T}\right) = Ce^{\frac{\mu-r}{\gamma\sigma}W_T} = \tilde{C}(S_T)^{\frac{\mu-r}{\gamma\sigma^2}}$
  - Delta-hedge X_T^* by holding $\Delta_t = \frac{\partial X_t^*}{\partial S_t} = \left(\frac{\mu-r}{\gamma\sigma^2}\right)\frac{X_t^*}{S_t}$ units of S_t
  - Replicate X_T^* by investing portion $\frac{\mu-r}{\gamma\sigma^2}$ of wealth in stocks

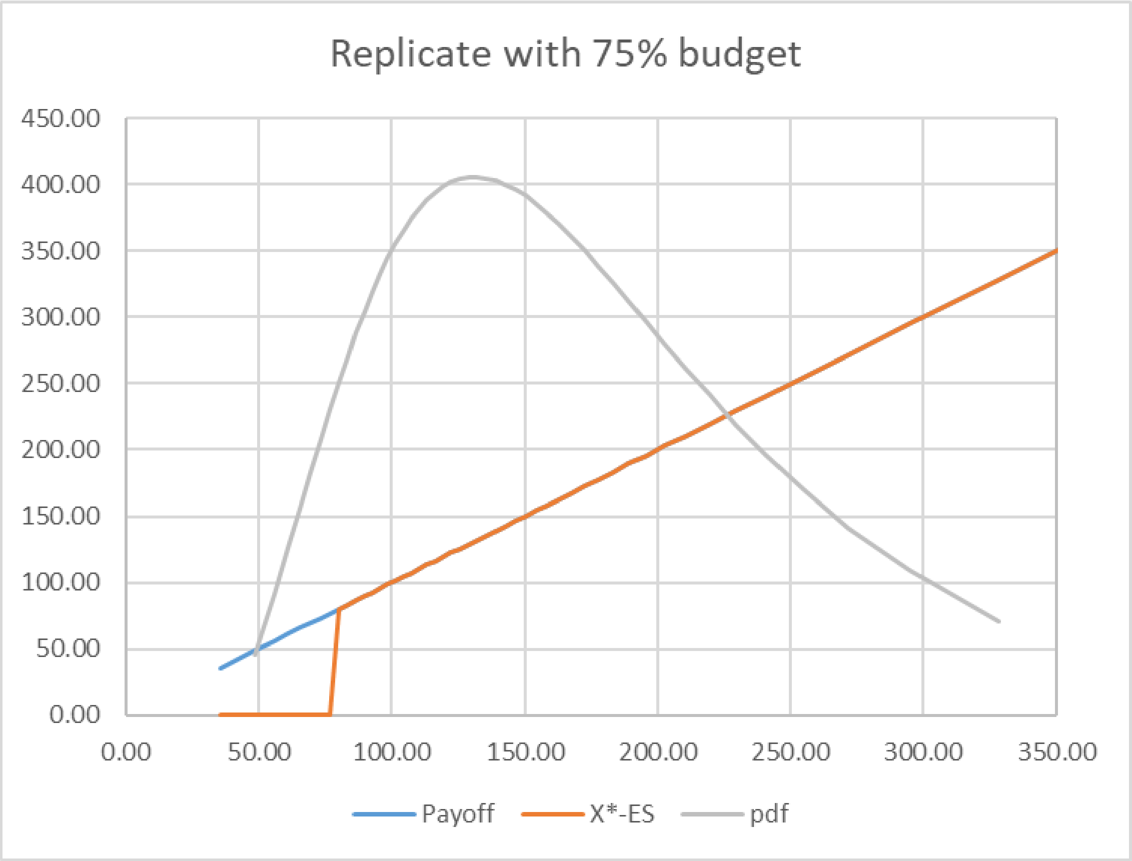
Minimise Underfunding

- Minimise underfunding w.r.t. (random) target Y_T
  - $U_{ES}(X_T) = min(X_T - Y_T , 0)$
- Consider $\mathbb{E}^{\mathbb{Q}}\left[\frac{Y_T}{B_T}\right] > X_0$: Y_T is more expensive than X_0
  - $U'(X_T) = \mathbb{I}_{X_T<Y_T}$ with inverse function
  - $I(y) = Y_T$ for $y \leq 1$ and $I(y) = 0$ for $y > 1$
- $X_T^* = I\left(\Lambda_0 \frac{\mathbb{Q}_T}{B_T}\right) = Y_T \mathbb{I}\left(\frac{\mathbb{Q}_T}{B_T} \leq C\right)$, solve C for budget X_0
  - Replicate Y_T except for "expensive" states: $\frac{\mathbb{Q}_T}{B_T} > C$

Minimise Underfunding (2)

- Optimal payoff, that minimises expected shortfall
- With only 75% budget

- See: Föllmer-Leukert (2000)



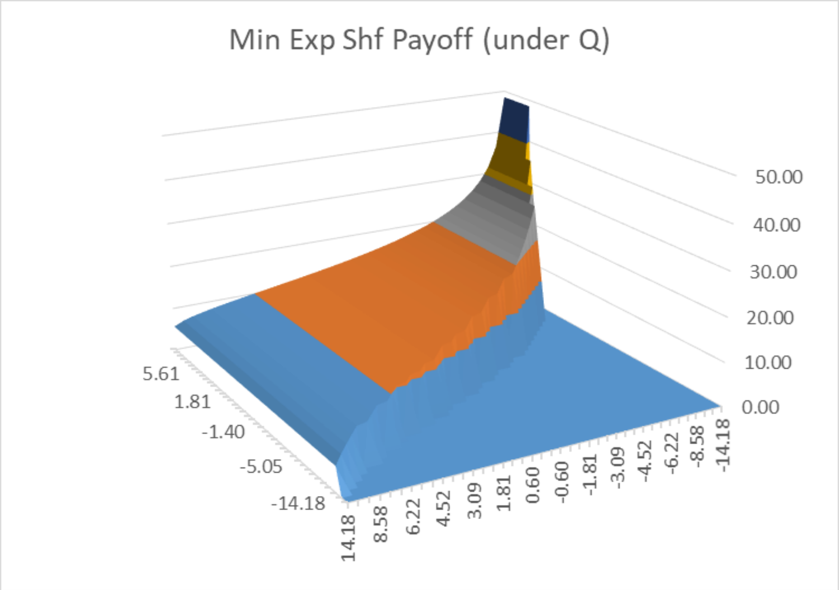
Minimise Underfunding of DB

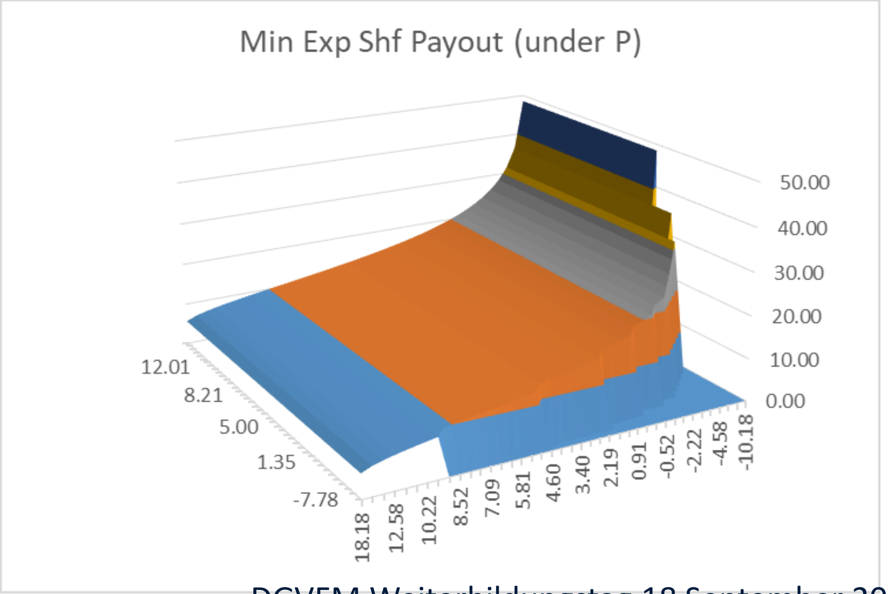
- Black-Scholes-Vasicek model
  - Stochastic stock return and stochastic (real) interest rates
- Target Y_T is a (real) annuity at retirement age T
  - Level of annuity depends on (real) interest rate
    - And mortality… which we ignore for now

- Optimal investment towards a Defined Benefit target Y_T within a Defined Contribution budget

Minimise Underfunding DB (2)

- Assume the NPV of all premiums only finances 50% of annuity market-value $\mathbb{E}^{\mathbb{Q}}[e^{-rT}Y_T]$ at t=0
  - Find optimal investment X_T that minimises expected shortfall: min(X_T-Y_T ,0)
  - Horizon of T=40 years
- Optimal investment strategy: achieves “success ratio” of over 95% (!)





Vector-AR model

- Many ALM models have multiple state-variables:
  - Nominal interest rates, stocks, real-estate, inflation, etc
- Suppose we have a state-vector Y_t that evolves as VAR model (or vector-OU model):
  - $dY_t = (\theta - AY_t)dt + \Sigma \cdot dW_t^{\mathbb{P}}$
- Change of measure: $dQ_t = Q_t\, \kappa' dW_t$ then:
  - $dY_t = \big((\theta + \Sigma\kappa) - AY_t\big)dt + \Sigma \cdot dW_t^{\mathbb{Q}}$
  - For constant κ, Q_t is a lognormal process

VAR model (2)

- We can solve the optimal investment for general utility *U()*:
  - Optimal wealth: $X_T^* = I\left(\Lambda_0 \frac{Q_T}{B_T}\right)$
  - Function I() is inverse of marginal utility U'()
  - Solve constant Λ_0 to fit budget constraint

- Investment strategy is given by "delta-hedging" current wealth: $X_t^* = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_T} I\left(\Lambda_0 \frac{Q_T}{B_T}\right)\right]$
  - Compute a "delta" w.r.t. each of the traded assets

VAR model (3)

- The model for the “Haalbaarheidstoets” (HBT) is a VAR-model
  - Prescribed by Dutch Central Bank (DNB) for pension funds
  - Stochastic model for stocks, interest rates, inflation
  - Model based on Koijen-Nijman-Werker, RFS (2010)

$$d\begin{bmatrix} X \\ \ln\Pi \\ \ln S \\ \ln P^{F0} \\ \ln P^{F\tau} \end{bmatrix} = \left(\begin{bmatrix} 0 \\ \delta_{0\pi} - \frac{1}{2}\sigma_\Pi'\sigma_\Pi \\ R_0 + \eta_S - \frac{1}{2}\sigma_S'\sigma_S \\ R_0 \\ R_0 + B^N(\tau)'\Sigma_X'\Lambda_0 - \frac{1}{2}B^{N\prime}\Sigma_X'\Sigma_X B^N \end{bmatrix} + \begin{bmatrix} -K & 0 \\ \delta_{1\pi}' & 0 \\ R_1' & 0 \\ R_1' & 0 \\ R_1' + B^N(\tau)'\Sigma_X'\Lambda_1 & 0 \end{bmatrix} \begin{bmatrix} X \\ \ln\Pi \\ \ln S \\ \ln P^{F0} \\ \ln P^{F\tau} \end{bmatrix} \right) dt + \begin{bmatrix} \Sigma_X' \\ \sigma_\Pi' \\ \sigma_S' \\ 0 \\ B^N(\tau)'\Sigma_X' \end{bmatrix} dZ_t$$

Conclusion

- With "martingale formulation" we are able to find general solution for optimal investment problem:
  - Holds for <u>any</u> (strictly concave) utility function *U()*
  - Holds for <u>any</u> pricing kernel $\mathbb{Q}_T/B_T$
    - Even with stochastic interest rates, stochastic volatility, etc
- VAR-models are explicitly solvable
  - Even for multiple assets, e.g. HBT model

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