Dividend problems for a Lévy insurance risk process

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Economic point of view

- The word *dividend* comes from the Latin word *dividendum* meaning *the thing which is to be divided* and has got sense of *portion of interest on a loan, stock, etc.*
- Dividends are usually defined as the distribution of earnings in real assets among the shareholders of the firm (in proportion to their ownership).
- Dividends are paid from the firm's after-tax income. For the recipient, dividends are considered regular income and are therefore fully taxable.
- There are two sides of dividends policies in the modern corporate firms. The first are managers of the firm (insiders), the second are shareholders (outsiders). The interest of management and shareholders may not coincide. This has important consequences for dividend policy. There is a suggestion that former typically prefer a low payout in order to pursue growth maximizing strategies or consume additional benefits, while letters generally wish for a high payout since this will force the management to incur the inspection of the capital markets for each new project undertaken.
- We focus in this talk on the maximizing the cumulant dividend payments (we look at it only from the point of view of beneficiaries).

Cramér-Lundberg model

Usually the surplus $X = \{X_t, t \geq 0\}$ of an insurance company is described by Cramér-Lundberg process:

$$X_t = x + pt - \sum_{k=1}^{N_t} C_k$$

where

C_k - i.i.d. claims with d.f. F

N_t - independent Poisson process with intensity λ

p - premium rate

For a classical risk process many optimal problems mentioned during this talk are analyzed in:
Hanspeter Schmidli
Stochastic Control in Insurance
Springer Verlag, London, (2008)

Lévy process

X_t - spectrally negative Lévy process which is not subordinator

will model reserves of insurance company (in the absence of dividend payments), that is:

X_t - process with independent and stationary increments without positive jumps

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Lévy-Khinchine formula:

$$Ee^{i\theta X_t} = e^{-\Psi(\theta)t}$$

where

$$\begin{aligned}\Psi(\theta) &= -ip\theta + \frac{\sigma^2}{2}\theta^2 + \int_{(-\infty,-1)} \left(1 - e^{i\theta x}\right) \Pi_X(dx) \\ &\quad + \int_{(-1,0)} \left(1 - e^{i\theta x} + i\theta x\right) \Pi_X(dx)\end{aligned}$$

and we assume that $\int_{(-1,0)}(1 \wedge x^2)\, \Pi_X(dx) < \infty$

De Finetti's problem

Let assume that $X_t \to \infty$ a.s.

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Controlled process:

$$U_t^\pi = X_t - D_t^\pi$$

where D_t^π denotes cumulative amount of dividends transferred up to time t

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$\pi = (\tau^\pi, D_t^\pi)$ - "dividend-liquidation" policy

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We observe process U^π up to $\tau^\pi \wedge \sigma^\pi$, where

$$\sigma^\pi = \inf\{t \geq 0 : U_t^\pi < 0\}$$

Impulse control

We assume that for $t < \sigma^\pi$

$$\Delta D_t^\pi := D_{t+}^\pi - D_t^\pi < U_t^\pi$$

When dividends are being paid it could be added fixed transaction cost $K > 0$ that are not transferred to the beneficiaries (case $K = 0$ means no transactions costs). In this case we assume additionally that:

$$\Delta D_t^\pi \geq K$$

In this case π is given by an increasing sequence $0 \leq T_1 \leq T_2 \leq \ldots$ of $\mathbf{F}$-stopping times representing the times at which a dividend payment is made and a sequence of positive $\mathcal{F}_{T_i}$-measurable random variables $J_i \geq K$ representing the sizes of the dividend payments. Then,

$$D_t^\pi = \sum_{k=1}^{N_t^\pi} J_k$$

where $N_t^\pi = \#\{k : T_k \leq t\}$

Value function

- cumulative discounted dividends received until the moment of ruin:

$$\mathcal{D}_\pi(x) = \mathbb{E}_x \left[\int_0^{\sigma^\pi \wedge \tau^\pi} \mathrm{e}^{-qt} (dD_t^\pi - K)_+ \right]$$

where

$$\int_0^t (dD_s^\pi - K)_+ = \begin{cases} D_t^\pi & \text{if } K = 0, \\ \sum_{s \le t} \mathbf{1}_{\{\Delta D_t^\pi > K\}} (\Delta D_t^\pi - K) & \text{if } K > 0 \end{cases}$$

and $\mathbb{E}_x$ denotes expectation when $X_0 = x$.

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- Gerber-Shiu penalty/reward function:

$$\mathcal{W}_w^\pi(x) = \mathbb{E}_x\left[\mathrm{e}^{-q(\sigma^\pi \wedge \tau^\pi)} w\left(U^\pi_{\sigma^\pi \wedge \tau^\pi}\right)\right]$$

where w is a penalty/reward function (for simplicity we assume that $w(0) = 0$)

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Value function:

$$v_\pi(x) = \mathcal{D}_\pi(x) + \mathcal{W}_w^\pi(x)$$

Goal

We want to find policy

$$\pi^* = (\tau^*, D^*)$$

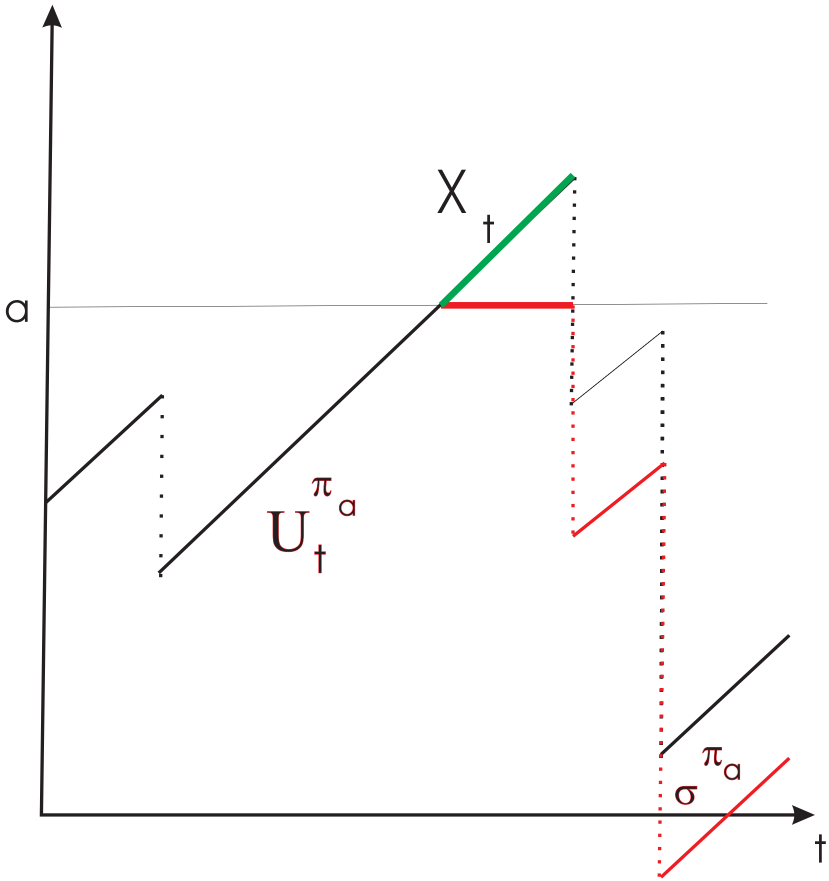
that maximizes

$$v_\pi(x) = \mathbb{E}_x \left[\int_0^{\sigma^\pi \wedge \tau^\pi} \mathrm{e}^{-qt} (dD_t^\pi - K)_+ \right] + \mathbb{E}_x \left[\mathrm{e}^{-q(\sigma^\pi \wedge \tau^\pi)} w\left(U^\pi_{\sigma^\pi \wedge \tau^\pi}\right) \right]$$

Then

$$v_*(x) = \sup_\pi v_\pi(x) = v_{\pi^*}(x)$$

Barrier strategy π_a for $K = 0$



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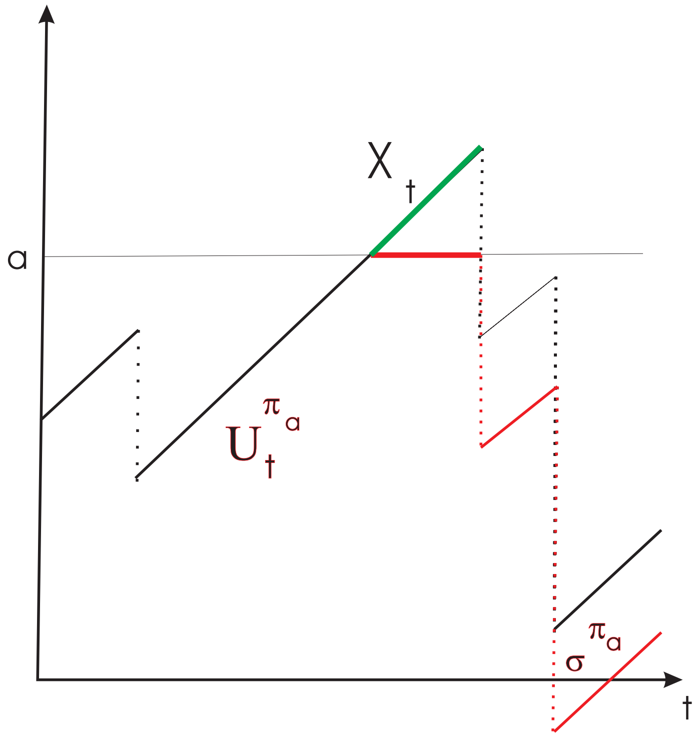


'If the barrier is too high, then we will wait too long for the risk process to hit the barrier and if we put the barrier too low then we derive the ruin too quickly.' We can then expect the existence of the 'optimal barrier'.

Local time at maximum

For the barier strategy at a and for $K = 0$:

$$D_t^{\pi_a} = a \vee \overline{X}_t - a, \qquad \text{where } \overline{X}_t = \sup_{s \le t} X_s$$

and

$$\{U_t^{\pi_a}, t \le \sigma^{\pi_a}; U_0^{\pi_a} = x\} \stackrel{D}{=} \{a - Y_t, t \le \sigma_a; Y_0 = a - x\}$$

where

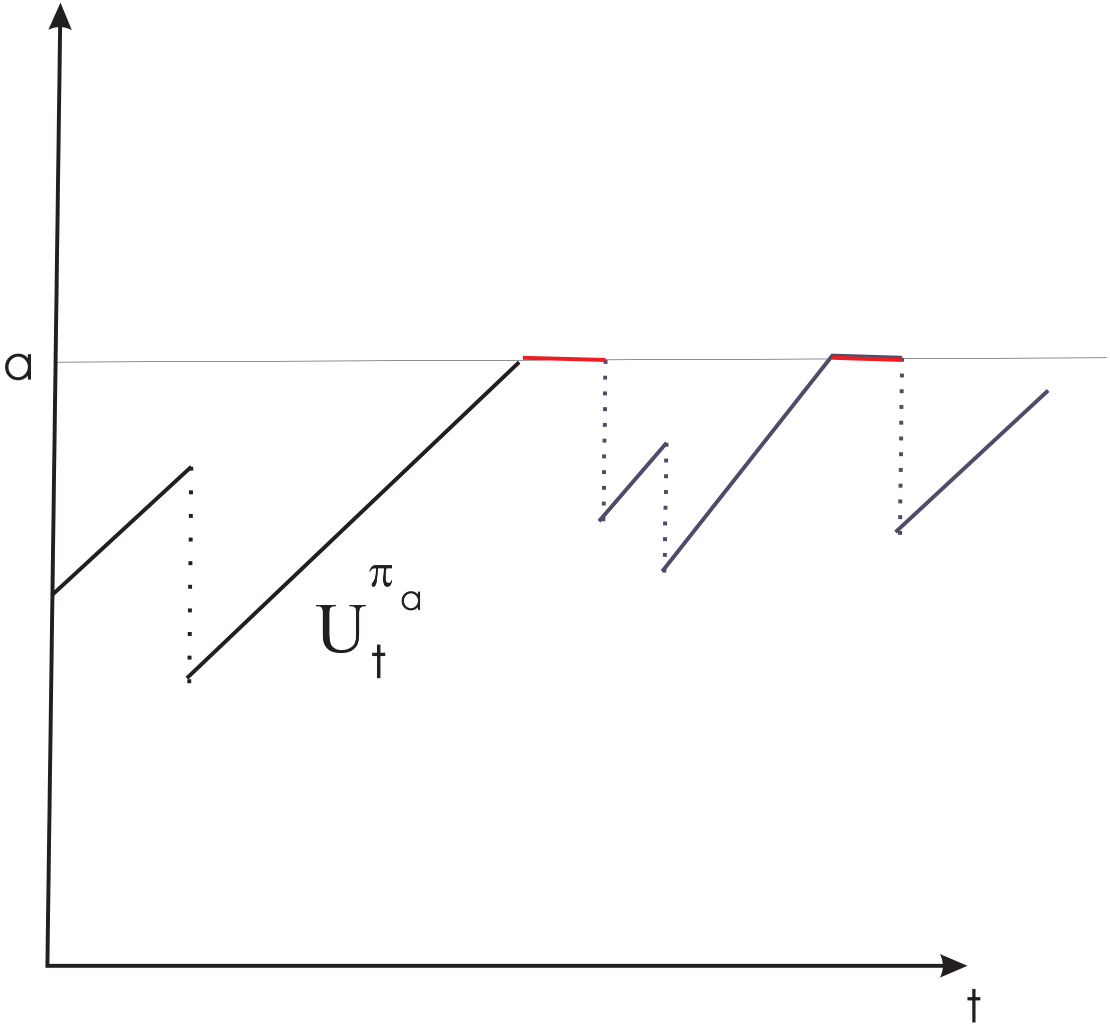
$$Y_t = (a \vee \overline{X}_t) - X_t$$

is reflected process at maximum and

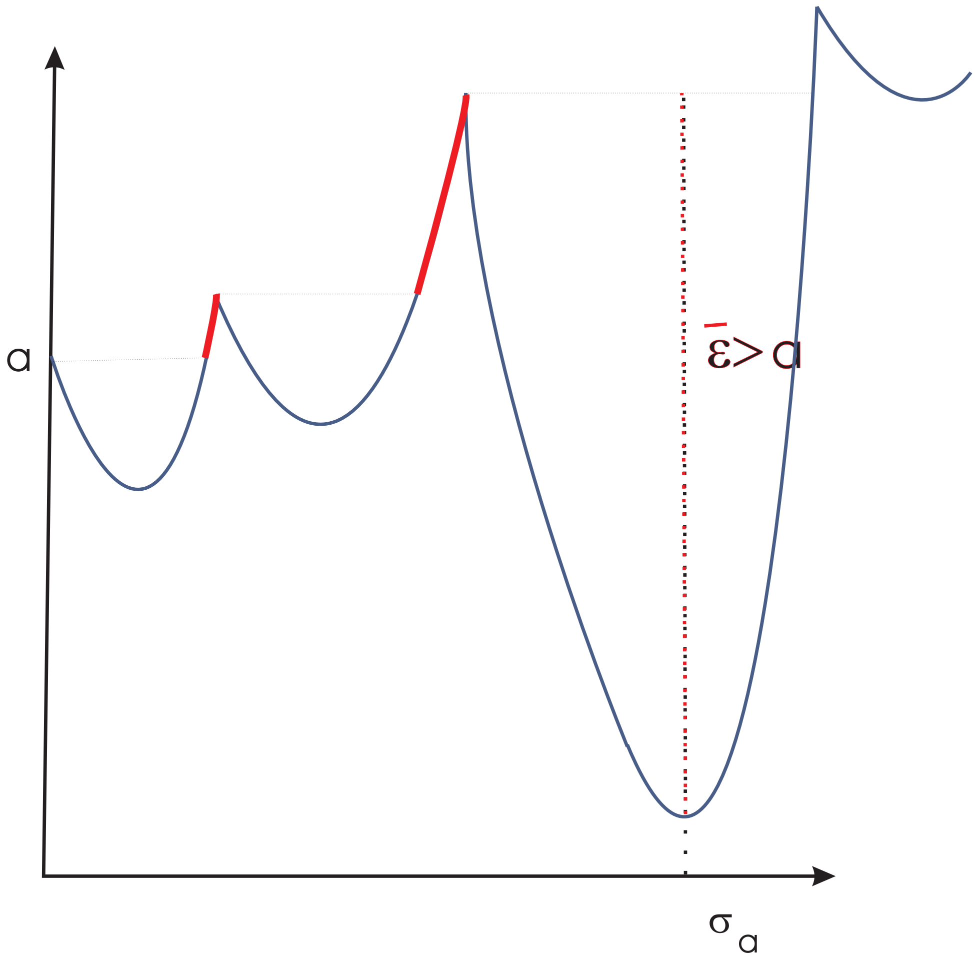
$$\sigma_a = \inf\{t > 0 : Y_t > a\}$$

is the first exit time of Y from interval $[0, a]$

Controlled process once again



Discounted local time



Bivariate subordinator

$$\mathcal{D}^{\pi_a}(x) = \int_0^{\sigma_a} e^{-qt} dD_t^{\pi_a} = \int_0^{\infty} e^{-qD_t^{\pi_a,-1}} \mathbf{1}_{(\sup_{s\le t} \bar{\epsilon}_s \le a)} dt$$
$$= \int_0^{\infty} e^{-q\xi_t} \mathbf{1}_{(\sup_{s\le t} \Delta\eta_s \le a)} dt$$

where $\Delta\eta_t = \eta_t - \eta_{t-}$ and $(\xi, \eta) = \{(\xi_t, \eta_t) : t \geq 0\}$ is a bivariate subordinator

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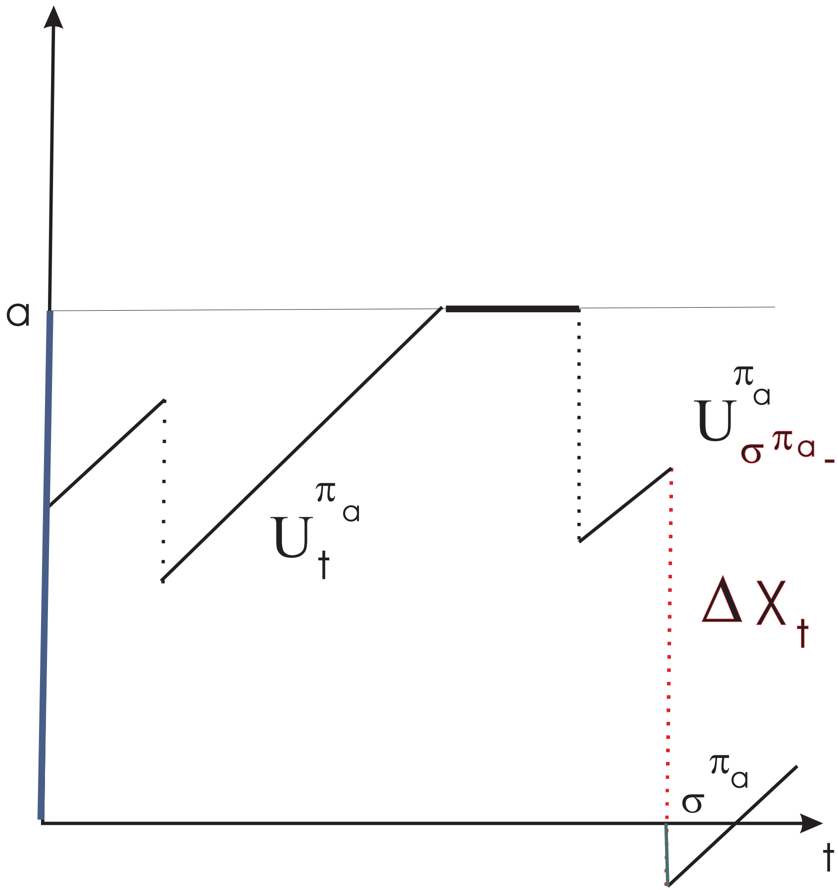
Theorem 1. (Kyprianou and Palmowski (2007)) For $n = 1, 2, 3, \ldots$ we have:

$$\mathbb{E}\left[\left(\int_0^\infty e^{-q\xi_t} \mathbf{1}_{(\sup_{s\le t} \Delta\eta_s \le a)} dt\right)^n\right] = n! \prod_{k=1}^n \frac{1}{\Lambda(qk) + \nu_{\Lambda(qk)}(a, \infty)}$$

where $\Lambda(q)$ is a Laplace exponent of ξ and $\nu_{\Lambda(q)}$ is a Lévy measure of η under the change of measure:

$$\frac{d\mathbb{P}_t^{\Lambda(q)}}{d\mathbb{P}_t} = e^{\Lambda(q)t - q\xi_t}$$

Barrier strategy π_a for $K = 0$



$$\mathcal{W}_w^{\pi_a}(x) = \mathbb{E}_x \left[\mathrm{e}^{-q\sigma^{\pi_a}} w(U_{\sigma^{\pi_a}}^{\pi_a}) \right]$$

Barrier strategy π_a for $K = 0$

$$\begin{aligned}
\mathcal{W}_w^{\pi_a}(x) &= \mathbb{E}_x\left[\mathrm{e}^{-q\sigma^{\pi_a}} w(U^{\pi_a}_{\sigma^{\pi_a}})\right] \\
&= \mathbb{E}_x\left[\sum_{t\geq 0} \mathrm{e}^{-qt} w(U^{\pi_a}_{t-} + \Delta X_t)\mathbf{1}_{\{t<\sigma^{\pi_a},\ \Delta X_t < U^{\pi_a}_{t-}\}}\right] \\
&= \int_0^a \int_y^\infty w(y-z)\Pi_X(dz)R^{(q)}(x,dy) = \int_0^a K_w(y)R^{(q)}(x,dy)
\end{aligned}$$

where $R^{(q)}(x,dy)$ is the resolvent of reflected at maximum process Y killed when exiting from the interval $[0,a]$:

$$\begin{aligned}
R^{(q)}(x,dy) &= \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(Y_t \in dy, t < \sigma_a)dt \\
&= \left[\frac{W^{(q)}(x)}{W^{(q)\prime}(a)} W^{(q)\prime}(a-y) - W^{(q)}(x-y)\right] dy \\
&\quad + \frac{W^{(q)}(x)}{W^{(q)\prime}(a)} W^{(q)}(0)\delta_a(dy)
\end{aligned}$$

Scale functions

Laplace exponent: $\psi(\theta)$:

$$\mathbb{E}[\mathrm{e}^{\theta X_t}] = \mathrm{e}^{t\psi(\theta)}$$

$\Phi(q)$ - greatest root of equation $\psi(\theta) = q$

First scaling function: $W^{(q)} : [0, \infty) \to [0, \infty)$:

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(y) dy = (\psi(\theta) - q)^{-1}, \qquad \theta > \Phi(q)$$

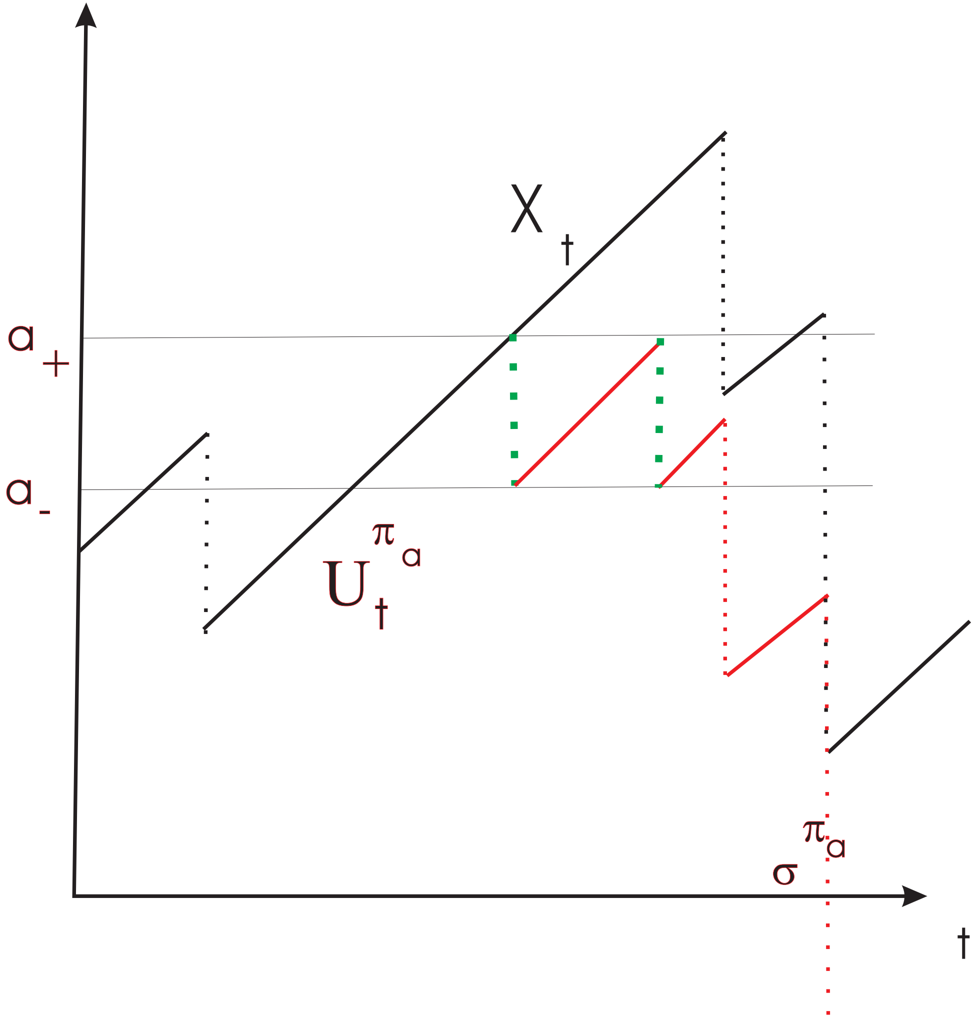
$W^{(q)}$ is differentiable (not necessary continuously) and

$$W(x) = W^{(0)}(x)$$

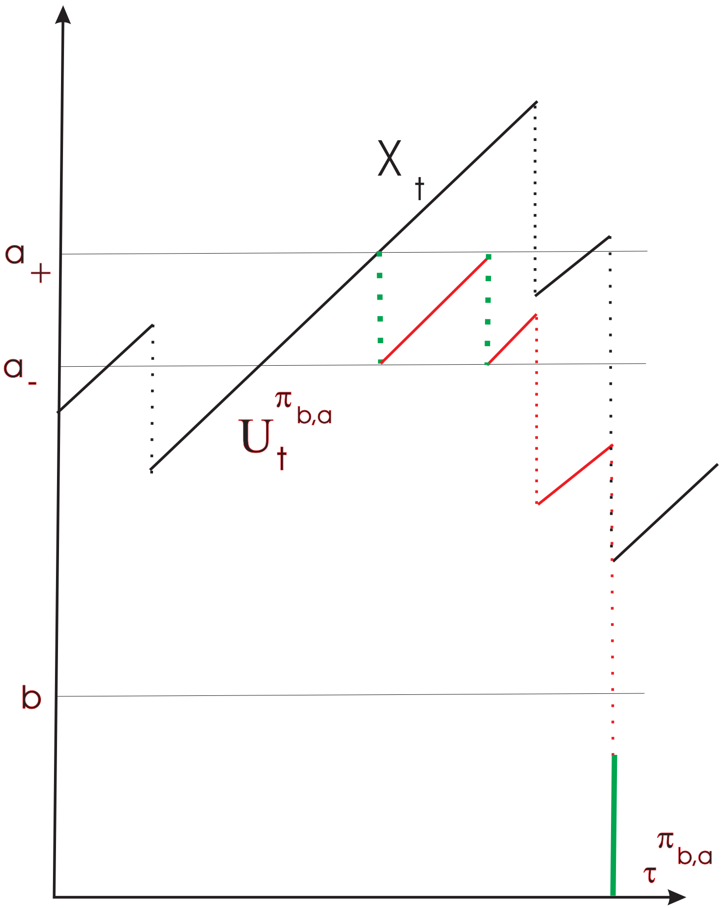
Second scaling function:

$$Z^{(q)}(y) = 1 + q \int_0^y W^{(q)}(z)\, dz$$

Barrier strategy π_a for $K > 0$

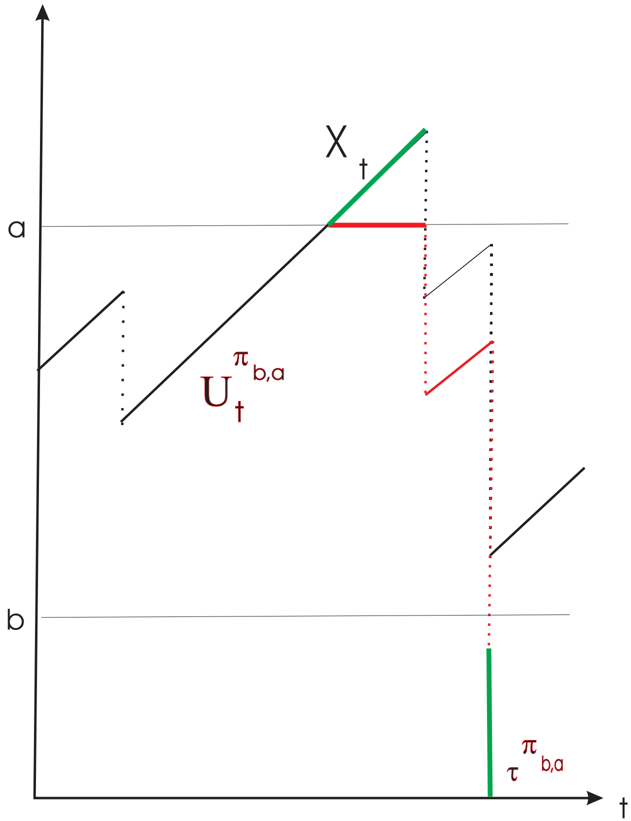


Barrier-liq. strategy $\pi_{b,a}$, $K > 0$



$\pi_{b,a}$ **for** $a = (a_-, a_+)$ **and** $b > 0$

Barrier-liq. strategy $\pi_{b,a}$, $K = 0$



$\pi_{b,a}$ for $a > b$ and $b > 0$ ('limiting case' when $a_+ = a_- = a$)

For simplicity we will use the same notation $\pi_{b,a}$ for both cases

Value function for $\pi_{b,a}$

Define:

$$\tilde{w}(x) = w(x-b), \qquad K_{\tilde{w}}(y) = \int_{-\infty}^{-y} \tilde{w}(y+z)\Pi_X(dz) < \infty$$

$$F_{\tilde{w}}(x) = -\int_0^x W^{(q)}(x-y)K_{\tilde{w}}(y)dy$$

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Theorem 2. (Avram, Palmowski and Pistorius (2013))

$$v_{\pi_{b,a}} := v_{b,a}(x) = \begin{cases} w(x) & x < b \\ W^{(q)}(x-b)G_b(a) + F_{\tilde{w}}(x-b) & x \in [b, a_+] \end{cases}$$

where

$$G_b(a) := \begin{cases} \dfrac{[\Delta a - K - \Delta F_{\tilde{w}}(a-b)}{\Delta W^{(q)}(a-b)} & \text{if } K > 0 \\ \dfrac{1 - F'_{\tilde{w}}(a-b)}{W^{(q)\prime}(a-b)} & \text{if } K = 0 \end{cases}$$

and $\Delta a = a_+ - a_-$, $\quad \Delta g(a-b) = g(a_+ - b) - g(a_- - b)$.

Optimality of $\pi_{b,a}$

We choose optimal barriers $a^* = (a^*_-, a^*_+)$ i b^* (for $K = 0$: $a^* = a^*_- = a^*_+$).
Let:

$$\mathcal{I}^*_w := \sup_{x>0} \mathcal{I}_w(x), \qquad \mathcal{I}_w(x) = \Gamma w(x) - qw(x)$$

where

$$\Gamma f(x) = \frac{\sigma^2}{2} f''(x) + pf'(x) + \int_{-\infty}^{0} \left[f(x+y) - f(x) - f'(x) y 1_{\{|y|<1\}} \right] \Pi_X(dy)$$

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Theorem 3. (APP (2013))
(i) If $\mathcal{I}^*_w \leq 0$, then it is optimal to stop immediately for all levels $x > 0$ of the reserves ($\tau_* \equiv 0$).

Assume that $\mathcal{I}^*_w > 0$. Then

(ii) π_{b^*,a^*} is the optimal strategy in the set of optimal strategies bounded by a^*_+;

(iii) if $(\Gamma v_{b^*,a^*} - q v_{b^*,a^*})(x) \leq 0$ for $x > a^*_+$, then π_{b^*,a^*} is optimal strategy and $v_* = v_{b^*,a^*}$. In particular, if Lévy measure admits a convex density then barrier-liquidation strategy is optimal.

HJB and new martinagels

The proof of Theorem 3 is based in verification theorem which says that optimal value function should satisfies HJB system:

$$\Gamma f(x) - qf(x) \leq 0$$
$$f(x) \geq f(y) + x - y - K$$
$$f(x) = w(x), \qquad \text{for } x < 0$$
$$f(x) \geq w(x), \qquad \text{for } x \geq 0$$

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In the proof we construct in fact new martingales, e.g for the first exit time $T_{b,a} = \inf\{t \geq 0 : X_t \notin [b, a]\}$ from interval $[b, a]$ processes:

$$\mathrm{e}^{-q(t \wedge T_{b,a})} W^{(q)} \left(X_{t \wedge T_{b,a}} - b \right)$$

$$\mathrm{e}^{-q(t \wedge T_{b,a})} F_{\tilde{w}} \left(X_{t \wedge T_{b,a}} - b \right)$$

are martingales

Particular cases

For $w(x) = \sum_{m=0}^{k} a_m x^m$ we have

$$F_{\tilde{w}}(x) = \sum_{m=0}^{k} a_m F_m(x-b)$$

where $F_0(x) = Z^{(q)}(x)$ and for $k \geq 1$:

$$F_k(x) = x^k + q\overline{W}^{(q,k)}(x) - \sum_{m=1}^{k} \binom{k}{m} \psi^{(m)}(0)\overline{W}^{(q,k-m)}(x)$$

and

$$\overline{W}^{(q,m)}(x) = \int_0^x (x-y)^m W^{(q)}(y)dy$$

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$$\overline{W}^{(q,m)}(x) = \int_0^x (x-y)^m W^{(q)}(y)dy$$

If $w(0) \neq 0$, then instead of $F_{\tilde{w}}$ we should take:

$$Q(x) = F_{\tilde{w}}(x) + w(0)\frac{\sigma^2}{2} W^{(q)\prime\prime}(x-b)$$

Azcue and Muler 2005

Cramér-Lundberg process with Gamma claims:

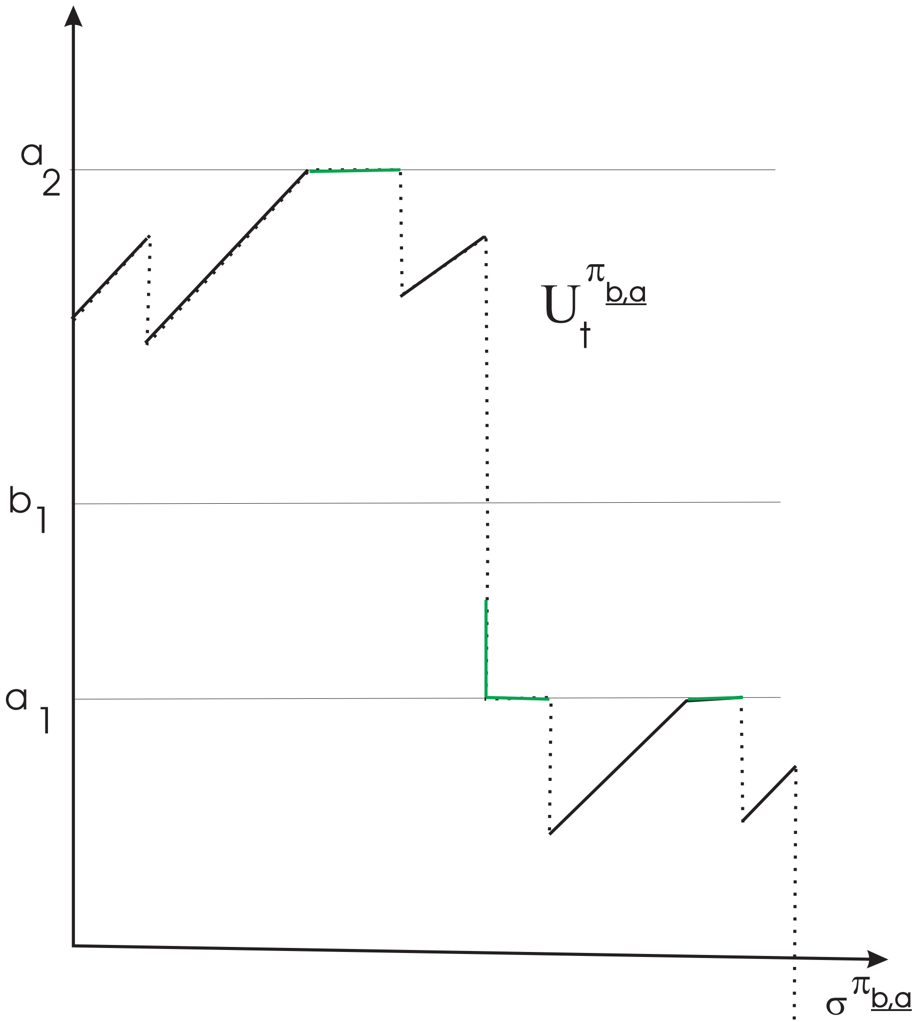
$$F(dx) = xe^{-x}dx,$$

discount intensity $q = 0.1$, Poisson intensity $\lambda = 10$,
premium rate $p = 2(1 + 0.07)\lambda$.

Then:

$$v_*(x) = \begin{cases} x + 2.119 & x \in [0, 1.803) \\ 0.0944e^{-1.4882x} - 9.431e^{-0.07953x} + 11.257e^{-0.03957x} & x \in [1.803, 10.22) \\ x + 2.456 & x \geq 10.22 \end{cases}$$

Band strategy $\pi_{\underline{b},\underline{a}}$



$\pi_{\underline{b},\underline{a}}$, $K = 0$

Theorem 4. (APP (2013)) Let $K = 0$. For $i \geq 1$ and $v_{\pi_{\underline{b},\underline{a}}} := v_{\underline{b},\underline{a}}(x)$:

$$v_{\underline{b},\underline{a}}(x) = \begin{cases} W^{(q)}(x - b_{i-1})G_{w_{i-1}}(a_i, b_{i-1}) + F_{w_{i-1}}(x - b_{i-1}) & x \in [b_{i-1}, a_i) \\ \\ v_{\underline{b},\underline{a}}(a_i -) + x - a_i & x \in [a_i, b_i) \end{cases}$$

where

$$F_{w_{i-1}}(x - b_{i-1}) = w'_{i-1}(b_{i-1}-)F_1(x) + w_{i-1}(b_{i-1}-)F_0(x) + F_{w_{i-1,0}}(x)$$

and

$$w_{i-1,0}(x) = v_{\underline{b},\underline{a}}(x - b_{i-1}) - v_{\underline{b},\underline{a}}(b_{i-1}) - (x - b_{i-1})v_{\underline{b},\underline{a}}(b_{i-1})$$

Optimality of $\pi_{\underline{b}^*,\underline{a}^*}$

Optimal bands:

a_i^* is determined by smooth fit condition of singular control:

$$0 = \lim_{x \downarrow a_i^*} v''_{\underline{a}^*,\underline{b}^*}(x) = \lim_{x \uparrow a_i^*} v''_{\underline{a}^*,\underline{b}^*}(x),$$

$b_i^* > 0$ is determined by smooth fit condition:

$$1 = \lim_{x \uparrow b_i^*} v'_{\underline{a}^*,\underline{b}^*}(x) = \lim_{x \downarrow b_i^*} v'_{\underline{a}^*,\underline{b}^*}(x)$$

if X has unbounded variation and determined by the continuous fit condition:

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if X has bounded variation.

Theorem 5. (APP (2013)) Band strategy $\pi_{\underline{b}^*,\underline{a}^*}$ is always optimal.

Exponential claims $\mathrm{Exp}(\mu)$

Let

$$X_t = x + pt - \sum_{k=1}^{N_t} C_k$$

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$$X_t = x + pt - \sum_{k=1}^{N_t} C_k$$

where $C_k = \mathrm{Exp}(\mu)$. Then

$$\begin{aligned} W^{(q)}(x) &= p^{-1}\left(A_+ \mathrm{e}^{q^+(q)x} - A_- \mathrm{e}^{q^-(q)x}\right), \\ Z^{(q)}(x) &= p^{-1} q \left(q^+(q)^{-1} A_+ \mathrm{e}^{q^+(q)x} - q^-(q)^{-1} A_- \mathrm{e}^{q^-(q)x}\right) \end{aligned}$$

where $A_\pm = \frac{\mu + q^\pm(q)}{q^+(q) - q^-(q)}$ and $q^+(q) = \Phi(q)$ and $q^-(q)$ solve equation $\psi(\theta) = q$:

$$q^\pm(q) = \frac{q + \lambda - \mu p \pm \sqrt{(q + \lambda - \mu p)^2 + 4pq\mu}}{2p}$$

Exponential claims $\mathrm{Exp}(\mu)$

Let

$$X_t = x + pt - \sum_{k=1}^{N_t} C_k$$

where $C_k = \mathrm{Exp}(\mu)$. Then

$$\begin{aligned} W^{(q)}(x) &= p^{-1}\left(A_+ \mathrm{e}^{q^+(q)x} - A_- \mathrm{e}^{q^-(q)x}\right), \\ Z^{(q)}(x) &= p^{-1}q\left(q^+(q)^{-1}A_+ \mathrm{e}^{q^+(q)x} - q^-(q)^{-1}A_- \mathrm{e}^{q^-(q)x}\right) \end{aligned}$$

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$$q^\pm(q) = \frac{q + \lambda - \mu p \pm \sqrt{(q + \lambda - \mu p)^2 + 4pq\mu}}{2p}$$

Let consider piecewise-linear Gerber-Shiu function:

$$w(x) = cx\mathbf{1}_{\{x<0\}} + (x - K)\mathbf{1}_{\{x>0\}}$$

Exponential claims $\mathrm{Exp}(\mu)$

Liquidation strategy is optimal when:

$$\mathcal{I}_w^* \leq 0$$

$$\Leftrightarrow$$

$$c \geq \max\{c_1, c_2\}$$

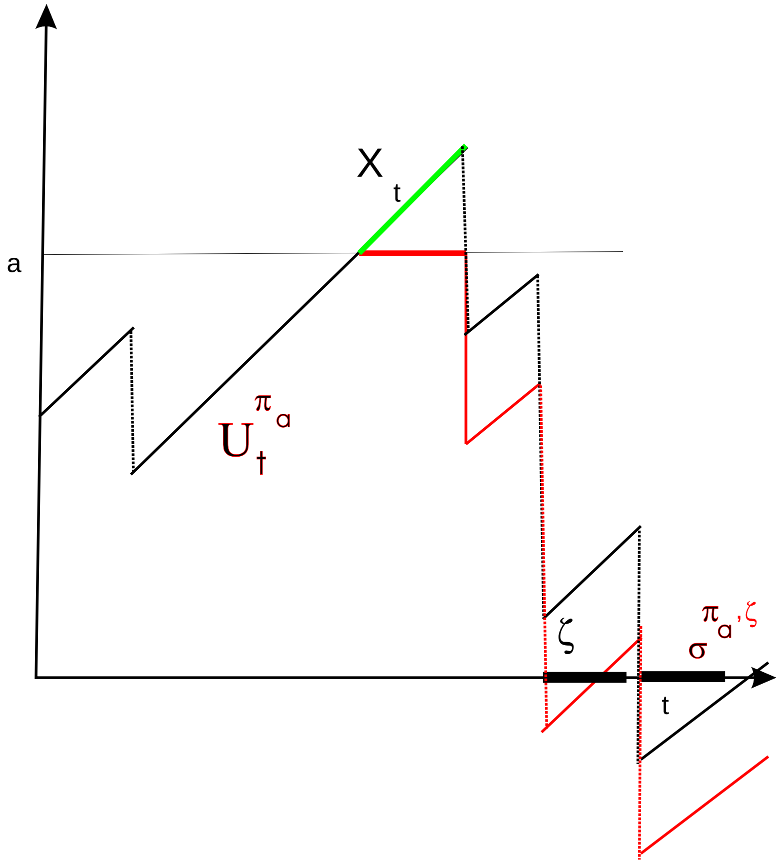
where

$$c_1 = (p\mu/\lambda + K(\mu + \frac{q\mu}{\lambda})), \qquad c_2 = 1 + K\mu + \frac{q}{\lambda}\exp\left\{(\frac{p\mu - \lambda}{q} + K\mu - 1)_+\right\}$$

If $c \in [c_1, c_2)$, then barrier-liquidation strategy π_{b^*,a^*} with $b^* > 0$ is optimal.

If $c < \min\{c_1, c_2\}$, then barrier strategy π_{a^*} is optimal (hence $b^* = 0$).

Parisian barrier startegy π_a



Parisian scale function

Everything remains the same if one takes (Czarna and Palmowski (2013)):

$$V^{(q)}(x) = e^{\Phi(q)x}\mathbb{P}_x^{\Phi(q)}(\tau^\zeta = \infty),$$

instead of $W^{(q)}(x)$, where

$$\tau^\zeta = \inf\{t > 0 : t - \sup\{s \leq t : X_s \geq 0\} \geq \zeta, X_t > 0\}$$

Theorem 6. (Czarna and Palmowski (2011), Loeffen, Czarna, Palmowski (2013)) For $x \geq 0$:

$$\mathbb{P}_x(\tau^\zeta < \infty) = 1 - \mathbb{E}X_1 \frac{\int_0^\infty W(x+z) z \mathbb{P}(X_\zeta \in \mathrm{d}z)}{\int_0^\infty z \mathbb{P}(X_\zeta \in \mathrm{d}z)}$$

Exponential claims $\mathrm{Exp}(\mu)$

If

$$X_t = x + pt - \sum_{k=1}^{N_t} C_k$$

where $C_k = \mathrm{Exp}(\mu)$, then

$$\mathbb{P}_x(\tau^\zeta < \infty) = \mathrm{e}^{(\frac{\lambda}{p}-\mu)x}\left(1 - \frac{p\mu - \lambda}{e^{-\lambda\zeta}p\mu + e^{-\lambda\zeta}\frac{\mu}{\zeta}D_2}\right)$$

for

$$D_2 = \int_0^{p\zeta}(p\zeta - t)\mathrm{e}^{-\mu t}\sqrt{\frac{\mu\lambda\zeta}{t}}I_1(2\sqrt{t\mu\lambda\zeta})\mathrm{d}t$$

C-L - numerical an.

We take the following parameters:
$w(x) = 0$ for $x \leq 0$, $\mu = 2, \lambda = 2, q = 0.1, p = 2.5$. Then

$$a^* = 3.78.$$

ζ	0.1	0.3	0.7	2
$a^{*,\zeta}$	3.54	3.09	2.40	0.84

Table 1: Optimal barrier for various Parisian delays.

x	2	5	10	50
$v(x)$	12.57	15.71	20.71	60.71
$v_{a^{*,\zeta}}(x)$	13.38	16.40	21.40	61.40

Table 2: Discounted cumulative dividends for classical and Parisian ruin for various initial capitals.

Let

$$\zeta = 0.3.$$

x_1	2.69	5.69	10.69	50.69
x	2	5	10	50
$v(x_1) = v_{a^*,\zeta}(x)$	13.38	16.40	21.40	61.40

Table 3: How much more initial capital is required to have the same amount of dividend payments for classical and Parisian ruin.

Brownian motion with drift

$$X_t = x + \sigma B_t + pt$$

For $x \geq 0$

$$\mathbb{P}_x(\tau^\zeta < \infty) = e^{-(2p\sigma^{-2})x} \frac{\Xi\left(\frac{p}{\sigma}\sqrt{\frac{\zeta}{2}}\right) - \frac{p}{\sigma}\sqrt{\frac{\zeta\pi}{2}}}{\Xi\left(\frac{p}{\sigma}\sqrt{\frac{\zeta}{2}}\right) + \frac{p}{\sigma}\sqrt{\frac{\zeta\pi}{2}}}$$

where

$$\Xi(x) = 2\sqrt{\pi}x\mathcal{N}(\sqrt{2}x) - \sqrt{\pi}x + e^{-x^2}$$

and $\mathcal{N}(\cdot)$ is a distribution function of standard normal random variable.

Brownian motion with drift

Let

$$X_t = \sigma B_t + pt$$

Then:

$$W^{(q)}(x) = \frac{1}{\sigma^2 \delta}[\mathrm{e}^{(-\omega+\delta)x} - \mathrm{e}^{-(\omega+\delta)x}]$$

where

$$\delta = \sigma^{-2}\sqrt{p^2 + 2q\sigma^2}$$

and

$$\omega = p/\sigma^2$$

Brownian motion with drift

$$a^* = \log \left| \frac{\delta + \omega}{\delta - \omega} \right|^{1/\delta}$$

for $\delta{=}\sigma^{-2}\sqrt{p^2 + 2q\sigma^2}$ and $\omega = p/\sigma^2$ and

$$a^{*,\zeta} = \frac{\sigma^2}{p_q} \log \left[\frac{\Psi\left(\frac{p_q}{\sigma}\sqrt{\frac{\zeta}{2}}\right) - \frac{p_q}{\sigma}\sqrt{\frac{\zeta\pi}{2}}}{\Psi\left(\frac{p_q}{\sigma}\sqrt{\frac{\zeta}{2}}\right) + \frac{p_q}{\sigma}\sqrt{\frac{\zeta\pi}{2}}} \left(1 - \frac{2p_q}{p_q - p}\right) \right]$$

for

$$\Psi(x) = 2\sqrt{\pi} x \mathcal{N}(\sqrt{2}x) - \sqrt{\pi}x + e^{-x^2}$$

and

$$p_q = \sqrt{p^2 + 2q\sigma^2}$$

Brownian mot. - numerical an.

We take the following parameters:
$w(x) = 0$ for $x \leq 0$, $\sigma = 2, p = 2.5$. Then

$$a^* = 5.28.$$

ζ	0.1	0.3	0.7	2
$a^{*,\zeta}$	4.48	3.89	3.12	1.17

Table 4: Optimal barrier for various Parisian delays.

x	2	5	10	50
$v(x)$	20.49	24.72	29.72	69.72
$v_{a^{*,\zeta}}(x)$	23.00	26.11	31.11	71.11

Table 5: Discounted cumulative dividends for classical and Parisian ruin for various initial capitals.

Brownian mot. - numerical an.

Let

$$\zeta = 0.3.$$

x_1	3.40	6.39	11.39	51.39
x	2	5	10	50
$v(x_1) = v_{a^*,\zeta}(x)$	23.00	26.11	31.11	71.11

Table 6: How much more initial capital is required to have the same amount of dividend payments for classical and Parisian ruin.

Randomized observation periods

Albrecher, Cheung and Thonhauser (2013) consider classical risk process and look at this process only at random observation times $\{T_k\}$ at which a lump sum dividend payment of size $x - b$ will take place if the current surplus level x exceeds the barrier level b ('discrete' version of the barrier strategy) and the process will be declared ruined if $x < 0$. Authors derive the integro-differential equation for the value function.

Bail-out setting

Harrison and Taylor SPA 6 1978 and Løkka and Zervos 2005 (for the Brownian motion)

$$\overline{\pi} = \{L_t^{\overline{\pi}}, R_t^{\overline{\pi}}, t \geq 0\}$$

non-decreasing $\mathbf{F}$-adapted processes describing the cumulative amount of paid dividends and cumulative amount of injected capital
The controlled risk process:

$$V_t^{\overline{\pi}} = X_t - L_t^{\overline{\pi}} + R_t^{\overline{\pi}}$$

Value function:

$$\overline{v}_*(x) = \sup_{\overline{\pi}} \mathbb{E}_x \left[\int_0^\infty \mathrm{e}^{-qt} dL_t^{\overline{\pi}} - \varphi \int_0^\infty \mathrm{e}^{-qt} dR_t^{\overline{\pi}} \right]$$

where $\varphi > 1$ is the cost per unit injected capital

Optimal policy

$$\overline{v}_a(x) = \begin{cases} \varphi(\overline{Z}^{(q)}(x) + \psi'(0^+)/q) + Z^{(q)}(x)\left[\frac{1-\varphi Z^{(q)}(a)}{qW^{(q)}(a)}\right] & 0 \le x \le a, \\ x - a + \overline{v}_a(a) & x > a \end{cases}$$

where

$$\overline{Z}^{(q)}(y) = \int_0^y Z^{(q)}(z)dz = y + q\int_0^y \int_0^z W^{(q)}(w)dwdz$$

Theorem 7 (Avram, Palmowski and Pistorius (2007)) Optimal strategy is a barrier strategy with the barrier

$$d^* = \inf\{a > 0 : G(a) \le 0\}$$

where $G(a) := [\varphi Z^{(q)}(a) - 1]W^{(q)\prime}(a) - \varphi q W^{(q)}(a)^2$

Capital injections

Scheer and Schmidli (2011), Kulenko and Schmidli (2008) maximise the discounted dividend payments minus the penalised discounted capital injections. Authors derive the Hamilton-Jacobi-Bellman equation for the problem and show that the optimal strategy is a barrier strategy.

DCF pricing

DCF - Discounted Cash Flow (Gajek and Kuciński (2011), only for Cramér-Lundberg process)
Value function:

$$\overline{v}_*(x) = \sup_{\overline{\pi},\sigma} \mathbb{E}_x \left[\int_0^\sigma \mathrm{e}^{-qt} dL_t^{\overline{\pi}} - \varphi \int_0^\sigma \mathrm{e}^{-qt} dR_t^{\overline{\pi}} \right]$$

Under classical assumptions the optimal strategy is to keep risk process non-negative until the first jump below level $-b < 0$

Pricing PZU This method produces price 8,03 bn EUR, the true market price in 2010 was: 7,97 bn EUR

Refraction

Process U solves equation:

$$U_t = X_t - \delta \int_0^t \mathbf{1}_{\{U_s > a\}}\, ds$$

Discounted cumulant dividends (Kyprianou and Loeffen (2010) and Gerber and Shiu (2006)):

$$E_x \int_0^{\sigma_a} e^{-qt} \mathbf{1}_{\{U_s > a\}}\, ds = -\int_0^{0 \wedge (x-a)} \mathbf{W}^{(q)}(z)\, dz + \frac{W^{(q)}(x) + \delta \mathbf{1}_{\{x>a\}} \int_x^a \mathbf{W}^{(q)}(x-y) W^{(q),\prime}(y)\, dy}{\phi(q) \int_0^\infty e^{-\phi(q) y} W^{(q),\prime}(y+a)\, dy}$$

where

$$\phi(q) = \sup\{\psi(\theta) - \delta\theta = q\}$$

and $\mathbf{W}^{(q)}$ and $\mathbf{Z}^{(q)}$ are the scale function associated with process $X_t - \delta t$

Taxes

$$U_t^{\gamma} = X_t - \int_0^t \gamma(\overline{X}_s)\, d\overline{X}_s$$

The process U^{γ} models the surplus process of an insurance company that pays out taxes according to a loss-carried-forward tax scheme, using a surplus-dependent rate $\gamma(\cdot)$. In other words, tax are collected when the company has recovered from its previous losses, i.e., is in a so-called profitable situation. Finally, note that when $\gamma(\cdot) = \gamma \in [0, 1]$, this model amounts to the situation studied in Albrecher et al. (2008) where the tax rate is constant, and when $\gamma = 1$, we retrieve the model where the company pays out as dividends any capital above its initial value $U^{\gamma} = x$ as in a risk model with a horizontal barrier strategy at level u (see e.g. Renaud and Zhou (2007)). Then:

$$E_x \int_0^{\sigma_a} e^{-qt} \gamma(\overline{X}_s)\, d\overline{X}_s = \int_x^{\infty} \exp\left\{ - \int_x^t \frac{W^{(q),\prime}(\overline{\gamma}(s))\, ds}{W^{(q)}(\overline{\gamma}(s))\, ds} \right\} \gamma(t)\, dt$$

where $\overline{\gamma}(y) = y - \int_x^y \gamma(s)\, ds$

Utility function

$$U_t^\pi = X_t - \int_0^t c_s \, ds$$

Value function:

$$v(x) = \sup \mathbb{E} \left(\int_0^\infty e^{-qt} m(c_t) dt \right)$$

for utility function m, e.g for:

$$m(x) = \frac{x^\alpha}{\alpha}, \qquad \alpha \in (0, 1)$$

Assume that the optimal strategy c^* is a function c^* of preset reserves. Then:

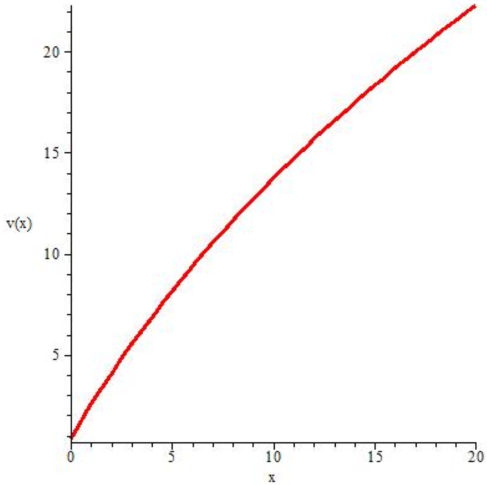
$$c^*(x) = (m')^{-1}(v_x)$$

C-L with Exp(μ)

For $m(x) = \frac{x^\alpha}{\alpha}$ the value function solves the following differential equation:

$$p v_{xx} + (\mu p - q - \lambda) v_x - \mu q v + \mu \frac{1-\alpha}{\alpha} v_x^{\frac{-\alpha}{1-\alpha}} - v_x^{\frac{-1}{1-\alpha}} v_{xx} = 0$$

$$v(0) = \frac{p}{q+\lambda} v_x(0) + \frac{1-\alpha}{\alpha(q+\lambda)} v_x(0)^{\frac{-\alpha}{1-\alpha}}$$



C-L with Exp(μ)

Theorem 8. (Baran and Palmowski (2013)) For large initial capital x we have:

$$v(x) \sim \left(\frac{\alpha}{2\alpha-1}\right)\left(\frac{1-\alpha}{\alpha\beta}\right)^{\frac{1-\alpha}{\alpha}} x^{\frac{2\alpha-1}{\alpha}}$$

$$v_x(x) \sim \left(\frac{1-\alpha}{\alpha q}\right)^{\frac{1-\alpha}{\alpha}} x^{\frac{-(1-\alpha)}{\alpha}}$$

$$c^*(x) \sim \left(\frac{1-\alpha}{\alpha q}\right)^{-\frac{1}{\alpha}} x^{\frac{1}{\alpha}}$$

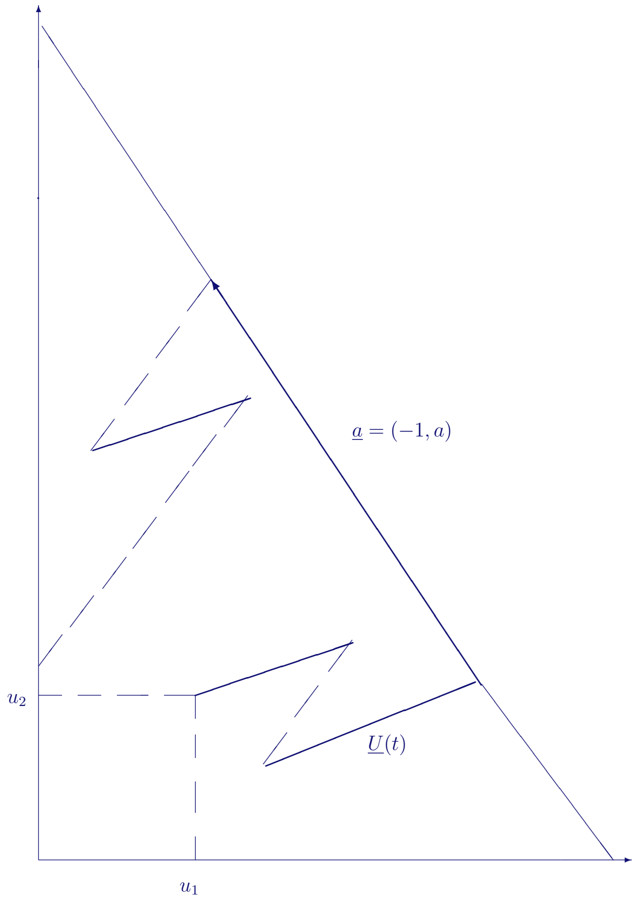
Two-dimensional risk process

Consider now a particular two-dimensional risk model in which two companies split the amount they pay out of each claim in fixed proportions (for simplicity we assume that they are equal), and receive premiums at rates p_1 and p_2, respectively (so-called proportional reinsurance). That is,

$$\underline{X}_t = (X_1(t), X_2(t)) = \left(u_1 + p_1 t - \beta_1 \sum_{i=1}^{N_t} C_i, u_2 + p_2 t - \beta_2 \sum_{i=1}^{N_t} C_i \right).$$

Without los of generality we will assume that $\beta_1 = \beta_2 = 1$ and $p_1 > p_2$

Two-dimensional risk process



Two-dimensional risk process

Controlled risk process:

$$\underline{U}_t = (U_1(t), U_2(t)) = \underline{X}_t - \underline{L}_t$$

where

$$\underline{L}(t) = \left(\delta_1 \int_0^t \mathbf{1}_{\{\underline{U}(t)\in\mathcal{B}\}},\ \delta_2 \int_0^t \mathbf{1}_{\{\underline{U}(t)\in\mathcal{B}\}}\right)$$

describes the two-dimensional linear drift at rate $\underline{\delta} = (\delta_1, \delta_2) > (0,0)$ which is subtracted from the increments of the risk process whenever it enter the fixed set:

$$\mathcal{B} = \{(x,y) : x, y \geq 0 \quad \text{and} \quad y \geq b - ax\}, \qquad a, b > 0.$$

The case $\underline{\delta} = \underline{p} - \underline{a}$ for $\underline{p} = (p_1, p_2)$ and $\underline{a} = (-1, a)$ corresponds to the reflecting the risk process at the line $y = b - ax$. Let

$$v_n(u_1, u_2) = v_n(\underline{u}) = \mathbb{E}_{\underline{u}}\left[(1,1) \cdot \int_0^\sigma e^{-qt}\, d\underline{L}(t)\right]^n$$

where $\sigma = \inf\{t \geq 0 : U_1(t)U_2(t) < 0\}$

Two-dimensional risk process

Theorem 9. (Czarna and Palmowski (2011))

$$\underline{p} \cdot \frac{\partial v_n}{\partial \underline{u}}(\underline{u}) - (\lambda + nq) v_n(\underline{u}) + \lambda \int_0^{\min(u_1,u_2)} v_n(\underline{u} - (1,1)v)\, dF(v) = 0$$

with the boundary conditions:

$$n\delta_0\, V_{n-1}(\underline{u}) = \underline{\delta} \cdot \left. \frac{\partial v_n}{\partial \underline{u}} \right|_{\underline{u} \in \mathcal{B}}, \qquad \underline{u} \in \mathcal{B}$$

$$\lim_{b \to \infty} v_n(\underline{u}) = 0, \qquad \underline{u} \in \mathcal{B}^{\mathrm{c}}$$

$$v_n(0, b) = 0$$

where $\delta_0 = \delta_1 + \delta_2$

Numerical analysis

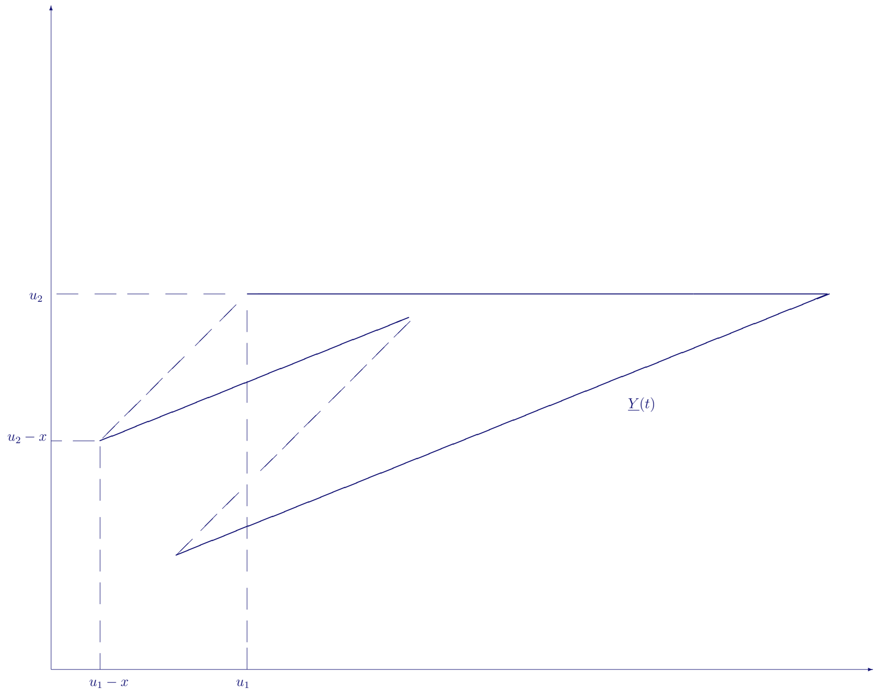
Assume that we have $\mathrm{Exp}(\mu)$ claims with $\mu = 2$ and that $p_1 = 4$, $p_2 = 3$, $\lambda = 1$, $q = 0.1$.

Note that always there exists optimal choice of linear barrier (choice of its upper left end $(0, b)$ and it slope a). This choice depends on the initial reserves (u_1, u_2). For $(u_1, u_2) = (1, 2)$ the optimal barrier is determined by $b = 14$ and $a = 0.1$ and for $(u_1, u_2) = (2, 3)$ the optimal barrier is determined by $b = 15$ and $a = 0.1$. This is contrast to the one-dimensional case where the choice of the barrier is given only via the premium rate and the distribution of the arriving claims.

a	b					
	6	8	14	15	20	28
0.1	19.85	27.20	34.95	34.93	32.48	25.89
0.2	16.33	24.31	33.82	34.19	33.32	28.03
0.5	11.76	17.74	28.98	30.01	32.54	31.21
1	7.22	11.40	21.35	22.59	27.17	30.07

Expected value of dividend payments depending on a and b for fixed $(u_1, u_2) = (1, 2)$.

Impulse control



THANK YOU

- for the Invitation !
- for Your Attention !