Closed-form solutions for GMABs

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Agenda

1. Introduction and motivation
2. Valuation model
3. Pricing of GMAB
4. Example
Introduction and motivation

What are Variable Annuities

- **Variable Annuities** (VA) are (deferred), fund-linked annuity and insurance products allowing guaranteed payments and participation in the financial markets at the same time.

- Examples for guaranteed payments include
  - roll-ups
  - ratchets

- Variable annuities are often referred to as GMxB, **Guaranteed Minimum Benefits** of type x:
  - GMDB (Death)
  - GMAB (Accumulation)
  - GMIB (Income)
  - GMWB (Withdrawal)
Introduction and motivation

Markets for Variable Annuities

- **Motivation**
  - Increasing life expectancy
  - Reduction of the state retirement pensions in several countries

- **Consequences**
  - VA - major success story in North American insurance market
  - Rapid growth of VA business in Japan - from $1.3 billion in 2001 to more than $216 billion in 2011 (assets under management)
  - Europe as the latest market for Variable Annuities

- **Risks:** financial, actuarial, behavioral
Introduction and motivation

Our contribution to existing literature

Most of the papers in the academic literature differentiate in: guarantees, financial and insurance models, consideration of policyholder behavior, pricing methods

- Milevsky and Posner [2001] GMDB
- van Haastrecht et al. [2009] GMAB
- Boyle and Hardy [2003], Marshall et al. [2010] GMIB
- Milevsky and Salisbury [2006], Jaimungal et al. [2012] GMWB
- Bauer et al. [2008], Bacinello et al. [2011] general framework for pricing GMxB’s.

Our contribution: Derivation of explicit solutions for the prices of GMAB products currently offered on the market in a hybrid model for insurance and market risks.
Valuation model

Financial market model

Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})\) be a filtered probability space with \(\mathbb{Q}\) being a risk-neutral measure

Financial market under \(\mathbb{Q}\) is described via Hull-White-Black-Scholes hybrid model with time-dependent volatility (HWBS\textsuperscript{tdv})

\[
    dr(t) = (\theta_r(t) - a_r(t))dt + \sigma_r dW^r(t),
\]

\[
    dY(t) = \left( r(t) - \frac{1}{2} \sigma_Y^2(t) \right) dt + \sigma_Y(t) dW^Y(t),
\]

where \(Y(t) = \ln(S(t)/S(0))\) and \(dW^r(t) dW^Y(t) = \rho dt\).
Random lifetime of a person aged $x$ at $t = 0$ is modeled as a stopping time $\tau_x$ of a counting process $\mathcal{N}_{x+t}(t)$ with corresponding mortality intensity $\lambda_{x+t}(t)$.

Two subfiltrations $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ and $\mathcal{H} = (\mathcal{H}_t)_{t \geq 0}$ of $\mathcal{F}$:

$$\mathcal{G}_t = \sigma(\lambda_{x+s}(s) : s \leq t), \quad \mathcal{H}_t = \sigma(\mathbb{1}_{\{\tau_x \leq s\}} : s \leq t).$$

Survival probability measured at time $t$ of a person at the age of $x + t$ to survive up to time $T$:

$$p_{x+t}(t, T|\mathcal{F}_t) := \mathbb{P}(\tau_x > T|\mathcal{F}_t) = \mathbb{E} \left[ e^{-\int_t^T \lambda_{x+s}(s)ds} | \mathcal{F}_t \right]$$
Valuation model

Insurance market model

- Compare the mortality intensity at time 0 with mortality intensity at time $t$
- Introduce mortality improvement ratio as

$$\xi_{x+t}(t) = \frac{\lambda_{x+t}(t)}{\lambda_{x+t}(0)}$$

Sample path for the mortality improvement ratio
Valuation model

Insurance market model

- We model $\xi_t$ as an extended Vasicek process
  
  $$d\xi(t) = k(e^{-\gamma t} - \xi(t))dt + \sigma_\xi dW_\xi(t)$$

- Initial mortality intensity is described via Gompertz model
  
  $$\lambda_{x+t}(0) = bc^{x+t}$$

  and is calibrated to the current life table.

- Future mortality intensity can be calculated as
  
  $$\lambda_{x+t}(t) = \lambda_{x+t}(0) \cdot \xi(t)$$

- Survival probabilities can be expressed as
  
  $$p_{x+t}(t, T|\mathcal{F}_t) = C_\lambda(t, T)e^{-D_\lambda(t, T)\lambda_{x+t}(t)},$$

  where $C_\lambda(t, T)$ and $D_\lambda(t, T)$ satisfy two ordinary differential equations, which can be solved analytically.
Pricing of GMAB

Definition

- GMAB provides a policyholder who is alive at the end of the accumulation period $T$ with a benefit $V(T)$, defined as

$$V(T) = \mathbb{1}_{\{\tau > T\}} \max(A(T), G(T))$$

- where $A(T)$ is the account value and $G(T)$ is one of the following guarantees
  - Return of premium: $G(T) = \text{IP}$
  - Roll-up $G(T) = \text{IP} \cdot e^{\delta T}$, where $\delta$ is a roll-up rate
  - Ratchet $G(T) = \max_{t_i < T} A(t_i)$

- The time 0 fair value of GMAB can be written as

$$V(0) = \mathbb{E}_Q \left[ e^{-\int_0^T r(s)ds} \mathbb{1}_{\{\tau > T\}} \max(A(T), G(T)) \right]$$
Explicit expression for $V(0)$ with $G(T) = IP \cdot e^{\delta T}$ can be derived:

$$V(0) = IP \cdot p_x(0, T) \cdot \Phi \left( \frac{\mu^S_{Y(T)} - \delta T}{\sigma^S_{Y(T)}} \right) + IP \cdot P^m(0, T) \cdot e^{\delta T} \cdot \Phi \left( \frac{\delta T - \mu^T_{Y(T)}}{\sigma^T_{Y(T)}} \right),$$

where $\Phi$ is standard normal distribution, $P^m(0, T)$ is mortality-adjusted zero-bond, $\mu^S_{Y(T)}, \sigma^S_{Y(T)}$ are the moments under equity measure $\mathbb{Q}^S$ and $\mu^T_{Y(T)}, \sigma^T_{Y(T)}$ are the moments under forward measure $\mathbb{Q}^T$. 
Pricing of GMAB

Ratchet guarantee

**Theorem**

Explicit expression for $V(0)$ with $G(T) = \max_{t_i < T} A(t_i)$ can be derived:

$$
V(0) = IP \cdot p_x(0, T) \cdot \left( \Phi_{n-1}(0; -\mu^S_{\Delta k} Y, \Sigma^S_{\Delta k} Y) \right. \\
\left. + \sum_{k=1}^{n-1} \left( \Phi_{n-1}(0; -\mu^S_{\Delta k} Y - \Sigma^S_{\Delta k} Y e_k, \Sigma^S_{\Delta k} Y) \right) \cdot e^{\mu^S_{\Delta n} Y + \frac{\left(\sigma^S_{\Delta n,k} Y \right)^2}{2}} \right),
$$

where $e_k$ is a unit vector, $\mu^S_{\Delta k} Y, \Sigma^S_{\Delta k} Y$ are mean vector and covariance matrix under equity measure $Q^S$ of

$$
\Delta_k Y := \{\Delta_{i,k} Y\}_{i \in \{1,\ldots,n\} \setminus \{k\}}, \quad \Delta_{i,k} Y := \{Y(t_k) - Y(t_i)\}_{i \in \{1,\ldots,n\} \setminus \{k\}}
$$
Example

Setup

- Type of the guarantee: single premium GMAB, $T = 20$ years
- Policyholder: male, 45 years old
- Mortality: German mortality table for 2007/2009
- Financial model: HWBS$^{tdv}$ calibrated to the market data as of 30/05/2012 (VSTOXX, EUR swap based yield curve and swaptions)

Ratchet step = 4 years

Roll-up rate = 2%
Sensitivities to product parameters

**Roll-up guarantee**

<table>
<thead>
<tr>
<th>Roll-up rate</th>
<th>GMAB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>102.49</td>
</tr>
<tr>
<td>1.5%</td>
<td>111.51</td>
</tr>
<tr>
<td>3.0%</td>
<td>125.64</td>
</tr>
</tbody>
</table>

**Ratchet guarantee**

<table>
<thead>
<tr>
<th>Ratchet step</th>
<th>GMAB</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 years</td>
<td>125.28</td>
</tr>
<tr>
<td>4 years</td>
<td>118.49</td>
</tr>
<tr>
<td>8 years</td>
<td>114.19</td>
</tr>
</tbody>
</table>
## Sensitivity analysis (roll-up)

- **Interest rates**
  
<table>
<thead>
<tr>
<th>IR</th>
<th>Roll-up 1</th>
<th>Roll-up 2</th>
<th>Roll-up 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity(^1)</td>
<td>-4.19%</td>
<td>-6.73%</td>
<td>-10.44%</td>
</tr>
</tbody>
</table>

- **Equity volatility**
  
<table>
<thead>
<tr>
<th>IR</th>
<th>Roll-up 1</th>
<th>Roll-up 2</th>
<th>Roll-up 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity</td>
<td>0.75%</td>
<td>0.98%</td>
<td>1.19%</td>
</tr>
</tbody>
</table>

\(^1\) based on a parallel shift of 0.01%
Conclusion

- HWBS$^{tdv}$ for the financial market
- 2-step approach for stochastic mortality modeling
- Explicit expressions for GMABs with different guarantee riders
- Calibration of the presented hybrid model
- Sensitivity analysis

Outlook

- Pricing of other guarantees (GMIB, GMDB)
- Incorporation of policyholder behavior risk (with Escobar, M., Ramsauer, F., Saunders, D., Zagst, R.)


