Closed-form solutions for GMABs

Mikhail Krayzler, Rudi Zagst, Bernhard Brunner

DGVFM
DEUTSCHE GESELLSCHAFT FÜR VERSICHERUNGS- UND FINANZMATHEMATIK e.V.

DAV/DGVFM: Scientific Day, 26.04.2013

Agenda

What are Variable Annuities

- **Variable Annuities** (VA) are (deferred), fund-linked annuity and insurance products allowing guaranteed payments and participation in the financial markets at the same time.
- Examples for guaranteed payments include
  - roll-ups
  - ratchets
- Variable annuities are often referred to as GMxB, **Guaranteed Minimum Benefits** of type x:
  - GM**D**B (Death)
  - GM**A**B (Accumulation)
  - GM**I**B (Income)
  - GM**W**B (Withdrawal)

Markets for Variable Annuities

- Motivation
  - Increasing life expectancy
  - Reduction of the state retirement pensions in several countries
- Consequences
  - VA - major success story in North American insurance market
  - Rapid growth of VA business in Japan - from $1.3 billion in 2001 to more than $216 billion in 2011 (assets under management)
  - Europe as the latest market for Variable Annuities
- Risks: financial, actuarial, behavioral

Our contribution to existing literature

Most of the papers in the academic literature differentiate in: guarantees, financial and insurance models, consideration of policyholder behavior, pricing methods

- Milevsky and Posner [2001] GMDB
- van Haastrecht et al. [2009] GMAB
- Boyle and Hardy [2003], Marshall et al. [2010] GMIB
- Milevsky and Salisbury [2006], Jaimungal et al. [2012] GMWB
- Bauer et al. [2008], Bacinello et al. [2011] general framework for pricing GMxB's.

Our contribution: Derivation of explicit solutions for the prices of GMAB products currently offered on the market in a hybrid model for insurance and market risks.

Financial market model

- Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ be a filtered probability space with $\mathbb{Q}$ being a risk-neutral measure
- Financial market under $\mathbb{Q}$ is described via Hull-White-Black-Scholes hybrid model with time-dependent volatility (HWBS^{tdv})

$$
\begin{aligned}
dr(t) &= (\theta_r(t) - a_r r(t))dt + \sigma_r dW^r(t), \\
dY(t) &= \left(r(t) - \frac{1}{2}\sigma_Y^2(t) \right) dt + \sigma_Y(t) dW^Y(t),
\end{aligned}
$$

where $Y(t) = ln(S(t)/S(0))$ and $dW^r(t)dW^Y(t) = \rho dt$.

Insurance market model

- Random lifetime of a person aged x at $t = 0$ is modeled as a stopping time τ_x of a counting process $N_{x+t}(t)$ with corresponding mortality intensity $\lambda_{x+t}(t)$
- Two subfiltrations $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ and $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ of $\mathbb{F}$:

$$\mathcal{G}_t = \sigma(\lambda_{x+s}(s) : s \leq t), \quad \mathcal{H}_t = \sigma(\mathbb{1}_{\{\tau_x \leq s\}} : s \leq t).$$

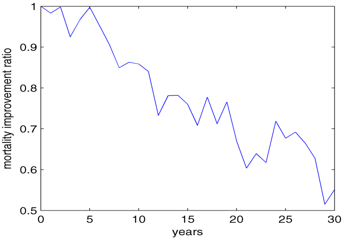
- Survival probability measured at time t of a person at the age of $x + t$ to survive up to time T

$$p_{x+t}(t, T|\mathcal{F}_t) := \mathbb{P}(\tau_x > T|\mathcal{F}_t) = \mathbb{E}\left[e^{-\int_t^T \lambda_{x+s}(s)ds} \middle| \mathcal{F}_t \right]$$

Insurance market model

- Compare the mortality intensity at time 0 with mortality intensity at time t
- Introduce **mortality improvement ratio** as

$$\xi_{x+t}(t) = \frac{\lambda_{x+t}(t)}{\lambda_{x+t}(0)}$$



Sample path for the mortality improvement ratio

Insurance market model

- We model ξ_t as an extended Vasicek process

$$d\xi(t) = k(e^{-\gamma t} - \xi(t))dt + \sigma_\xi dW^\xi(t)$$

- Initial mortality intensity is described via Gompertz model

$$\lambda_{x+t}(0) = bc^{x+t}$$

  and is calibrated to the current life table.

- Future mortality intensity can be calculated as

$$\lambda_{x+t}(t) = \lambda_{x+t}(0) \cdot \xi(t)$$

- Survival probabilities can be expressed as

$$p_{x+t}(t, T|\mathcal{F}_t) = C_\lambda(t, T)e^{-D_\lambda(t,T)\lambda_{x+t}(t)},$$

  where $C_\lambda(t, T)$ and $D_\lambda(t, T)$ satisfy two ordinary differential equations, which can be solved analytically.

Definition

- GMAB provides a policyholder who is alive at the end of the accumulation period T with a benefit $V(T)$, defined as

$$V(T) = \mathbb{1}_{\{\tau > T\}} max(A(T), G(T))$$

- where $A(T)$ is the account value and $G(T)$ is one of the following guarantees
  - Return of premium: $G(T) = \text{IP}$
  - Roll-up $G(T) = \text{IP} \cdot e^{\delta T}$, where δ is a roll-up rate
  - Ratchet $G(T) = \max_{t_i < T} A(t_i)$

- The time 0 fair value of GMAB can be written as

$$V(0) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds} \mathbb{1}_{\{\tau > T\}} max(A(T), G(T))\right]$$

Roll-up guarantee

Theorem

Explicit expression for $V(0)$ with $G(T) = IP \cdot e^{\delta T}$ can be derived:

$$\begin{aligned} V(0) &= IP \cdot p_x(0,T) \cdot \Phi\left(\frac{\mu^S_{Y(T)} - \delta T}{\sigma^S_{Y(T)}}\right) \\ &+ IP \cdot P^m(0,T) \cdot e^{\delta T} \cdot \Phi\left(\frac{\delta T - \mu^T_{Y(T)}}{\sigma^T_{Y(T)}}\right), \end{aligned}$$

where Φ is standard normal distribution, $P^m(0,T)$ is mortality-adjusted zero-bond, $\mu^S_{Y(T)}, \sigma^S_{Y(T)}$ are the moments under equity measure $\mathbb{Q}^S$ and $\mu^T_{Y(T)}, \sigma^T_{Y(T)}$ are the moments under forward measure $\mathbb{Q}^T$.

Ratchet guarantee

Theorem

Explicit expression for $V(0)$ with $G(T) = \max_{t_i < T} A(t_i)$ can be derived:

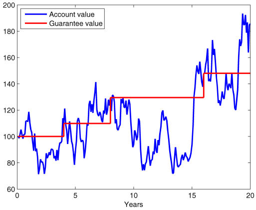
$$\begin{aligned} V(0) &= IP \cdot p_x(0,T) \cdot \Bigg(\Phi_{n-1}(0; -\mu^S_{\Delta_k Y}, \Sigma^S_{\Delta_k Y}) \\ &+ \sum_{k=1}^{n-1} \Big(\Phi_{n-1}(0; -\mu^S_{\Delta_k Y} - \Sigma^S_{\Delta_k Y} e_k, \Sigma^S_{\Delta_k Y}) \Big) \cdot e^{\mu^S_{\Delta_{n,k} Y} + \frac{\left(\sigma^S_{\Delta_{n,k} Y}\right)^2}{2}} \Bigg), \end{aligned}$$

where e_k is a unit vector, $\mu^S_{\Delta_k Y}, \Sigma^S_{\Delta_k Y}$ are mean vector and covariance matrix under equity measure $\mathbb{Q}^S$ of

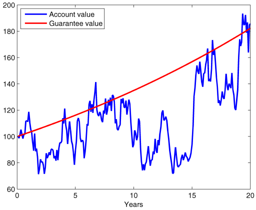
$$\Delta_k Y := \{\Delta_{i,k} Y\}_{i \in \{1,\dots,n\} \setminus \{k\}}, \quad \Delta_{i,k} Y := \{Y(t_k) - Y(t_i)\}_{i \in \{1,\dots,n\} \setminus \{k\}}$$

Setup

- Type of the guarantee: single premium GMAB, $T = 20$ years
- Policyholder: male, 45 years old
- Mortality: German mortality table for 2007/2009
- Financial model: HWBS^{tdv} calibrated to the market data as of 30/05/2012 (VSTOXX, EUR swap based yield curve and swaptions)



Ratchet step = 4 years

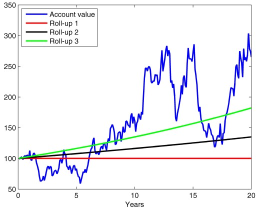


Roll-up rate = 2%

Example

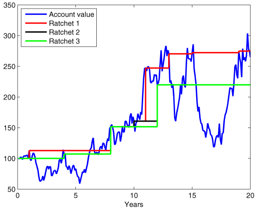
Sensitivities to product parameters

- Roll-up guarantee



Roll-up rate	GMAB
0%	102.49
1.5%	111.51
3.0%	125.64

- Ratchet guarantee



Ratchet step	GMAB
2 years	125.28
4 years	118.49
8 years	114.19

Sensitivity analysis (roll-up)

- Interest rates

IR	Roll-up 1	Roll-up 2	Roll-up 3
Sensitivity[1]	-4.19%	-6.73%	-10.44%

- Equity volatility

IR	Roll-up 1	Roll-up 2	Roll-up 3
Sensitivity	0.75%	0.98%	1.19%

[1] based on a parallel shift of 0.01%

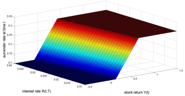
DGVFM
DEUTSCHE GESELLSCHAFT FÜR VERSICHERUNGS- UND FINANZMATHEMATIK e.V.

Conclusion

- HWBS^{tdv} for the financial market
- 2-step approach for stochastic mortality modeling
- Explicit expressions for GMABs with different guarantee riders
- Calibration of the presented hybrid model
- Sensitivity analysis

Outlook

- Pricing of other guarantees (GMIB, GMDB)
- Incorporation of policyholder behavior risk (with Escobar, M., Ramsauer, F., Saunders, D., Zagst, R.)



DGVFM
DEUTSCHE GESELLSCHAFT FÜR VERSICHERUNGS- UND FINANZMATHEMATIK e.V.

Bibliography

A. Bacinello, P. Millossovich, A. Olivieri, and E. Pitacco. Variable annuities: a unifying valuaion approach. *Insurance: Mathematics and Economics*, 49:285 – 297, 2011.

D. Bauer, A. Kling, and J. Ruß. A universal pricing framework for guaranteed minimum benefits in variable annuieties. *ASTIN Bulletin*, 38:621 – 651, 2008.

P. Boyle and M. Hardy. Guaranteed annuity options. *ASTIN Bulletin*, 33:125 – 152, 2003.

S. Jaimungal, R. Donnelly, and D. Rubisov. Valuing GWBs with stochastic interest rates and volatility. *Working paper*, 2012.

C. Marshall, M. Hardy, and D. Saunders. Valuation of a guaranteed minimum income benefit. *North American Actuarial Journal*, 14(1):38 – 58, 2010.

M. Milevsky and S. Posner. The titanic option: Valuation of the guaranteed minimum death benefit in variable annuities and mutual funds. *The Journal of Risk and Insurance*, 68(1):93 – 128, 2001.

M. Milevsky and T. Salisbury. Financial valuation of guaranteed minimum withdrawal benefits. *Insurance: Mathematics and Economics*, 38(1):21 – 38, 2006.

A. van Haastrecht, R. Lord, A. Pelsser, and D. Schrager. Pricing long-dated insurance contracts with stochastic volatility and stochastic interest rates. *Insurance: Mathematics and Economics*, 45(3):436 – 448, 2009.