DGVFM DEUTSCHE GESELLSCHAFT FÜR VERSICHERUNGS- UND FINANZMATHEMATIK e.V.

DGVFM-Jahrestagung, Wissenschaftstag 2013, Berlin 26.04.2013

DAV DEUTSCHE AKTUARVEREINIGUNG e.V.

Save for Bad Times or Consume as Long as You Have?

Worst-Case Optimal Life-Time Consumption

Ralf Korn (TU Kaiserslautern, Fraunhofer ITWM, EI-QFM)

Fraunhofer ITWM

EI-QFM

(CM)² Center for Mathematical + Computational Modelling

Technische Universität Kaiserslautern

Based on joint work with:

- Sascha Desmettre (Fraunhofer ITWM, Kaiserslautern)
- Frank Seifried (TU Kaiserslautern)

A preliminary remark:

- Assume that possibly during your life-time you may experience a once-in-a-life-time catastrophy (World war, Great Depression, Subprime Crisis, …)
- Assume that there is no probability to judge how likely it is to experience it ("Knightian Uncertainty")
- Assume that at that event you loose an a prioi unknown fraction of your wealth invested in stocks

Would you build up **extra savings** for the bad times

or

would you **spend more** as long as you have it ?

Ingredient I: Optimal Life-Time Consumption – 1

Problem:

How to distribute your consumption (as a fraction of your wealth) over your life-time?

References: Optimal Life-Time Consumption Problems are treated in

- Mossin (1968), Samuelson (1969) – Discrete-time model
- Merton (1969) – Continuous-time model
- Foldes (1990), Kramkov, Schachermayer (1999) – General utility function and semimartingale prices
- Davis, Norman (1990) – Continuous-time with proportional transaction costs
- Korn (1998) – Continuous-time with general transaction costs
- + many more

Ingredient I: Optimal Life-Time Consumption – 2

The Merton problem

In a Black-Scholes-type market with a money market account and a stock given by

$$dP_0(t) = P_0(t)\, r\, dt, \qquad P_0(0) = 1, \qquad \text{“MMA”}$$

$$dP_1(t) = P_1(t)\big((r+\lambda)dt + \sigma dW(t)\big),\ i = 1,.., n, \qquad P_1(0) = p_1, \qquad \text{“Stock”},$$

Goal: Find a pair of portfolio/consumption process $\left(\hat{\pi}(.),\hat{c}(t)\right)$ solving

$$\max_{(\pi,c)\in A(x)} E\left(\int_0^\infty e^{-\delta t}\frac{1}{1-\gamma}\left(c(t)X^{\pi,c}(t)\right)^{1-\gamma}dt\right)$$

with wealth process $dX^{\pi,c}(t) = X^{\pi,c}(t)\Big[\big(r+\pi(t)\lambda - c(t)\big)dt + \pi(t)\sigma dW(t)\Big], X^{\pi,c}(0) = x.$

The Merton solution (Merton (1969)):

(1) $\hat{\pi}(t) = \dfrac{\lambda}{\gamma\sigma^2}, \hat{c}(t) = A^{-1/\gamma}, \quad v_0(x) = A\frac{1}{1-\gamma}x^{1-\gamma}, \quad A = \left(\frac{\delta-(1-\gamma)\psi}{\gamma}\right)^{-\gamma}, \psi = r + \frac{1}{2}\dfrac{\lambda^2}{\gamma\sigma^2}$

where $v_0(x) := \sup_{(\pi,c)\in A(x)} E^x\left(\int_0^\infty e^{-\delta t}\frac{1}{1-\gamma}\left(c(t)X^{\pi,c}(t)\right)^{1-\gamma}dt\right)$ is the value function.

Ingredient II: Crash-Modelling and Worst-Case Approach – 1

Hua, Wilmott (1997), K., Wilmott (2002), K., Menkens (2005), K. Steffensen (2007), Seifried (2010), …

1. Hua & Wilmott (1997): "Number and size of crashes until a given time horizon are bounded"
$\Rightarrow$ **no** probabilistic assumptions on height, number and time of crashes.

2. K. & Wilmott (2001) "Determine worst-case bounds for optimal investment".

For simplicity: At most one crash in $[0, T]$ with maximum height $k^* < 1$.

$$dP_0(t) = P_0(t) r\, dt, \qquad P_0(0) = 1, \qquad \text{"MMA"}$$

$$dP_1(t) = P_1(t)\big((r+\lambda)dt + \sigma dW(t)\big), \qquad P_1(0) = p, \qquad \text{"Stock in normal times"}$$

At crash time: Stock price falls by a fraction $k \in [0, k^*]$

Consequence: Wealth process $X^\pi(t)$ at crash time:

$$X^\pi(t-) = \big(1-\pi(t)\big)X^\pi(t-) + \pi(t)X^\pi(t-)$$

$$\Rightarrow \big(1-\pi(t)\big)X^\pi(t-) + \pi(t)X^\pi(t-)(1-k) = X^\pi(t-)\big(1-\pi(t)k\big) = X^\pi(t)$$

Ingredient II: Crash-Modelling and Worst-Case Approach – 2

Aim: Find the best uniform **worst-case bound**, i.e. solve

$$\text{(WP)} \qquad \sup_{\pi(.)\in A(x)} \; \inf_{0\le t\le T,\, 0\le k\le k^*} E\left(\frac{1}{1-\gamma}\left(X^{\pi}(T)\right)^{1-\gamma} \right)$$

with final wealth $X^{\pi}(T) = \left(1-\pi(t)k\right)\tilde{X}^{\pi}(T)$ in case of a crash of size *k* at time *t*.

Ingredient II: Crash-Modelling and Worst-Case Approach – 3

Two extreme strategies (for log-utility, i.e. γ=0):

i) $\pi(t) \equiv 0$: **"Play safe"**

$\Rightarrow$ Worst-case szenario: no crash (!) with worst-case bound

$$WCB_0 = E\left(ln\left(X^0(T)\right)\right) = ln(x) + rT$$

ii) $\pi(t) \equiv \pi^* := \frac{b-r}{\sigma^2}$: **"Optimal investment in the crash-free world "**

$\Rightarrow$ worst-case szenario: crash of maximum height k* (at any time !) with worst-case bound

$$WCB_{\pi^*} = E\left(ln\left(X^{\pi^*}(T)\right)\right) = ln(x) + rT + \tfrac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 T + ln(1-\pi^* k^*)$$

Insights:

- it depends on *T* which strategy performs better
- a constant portfolio **cannot be** optimal
- Strategy i) is to conservative to be good if no crash happens, strategie ii) is too risky to perform well if a high crash appears

  $\Rightarrow$ ***an optimal strategy should balance this out*** !

Ingredient II: Crash-Modelling and Worst-Case Approach – 4

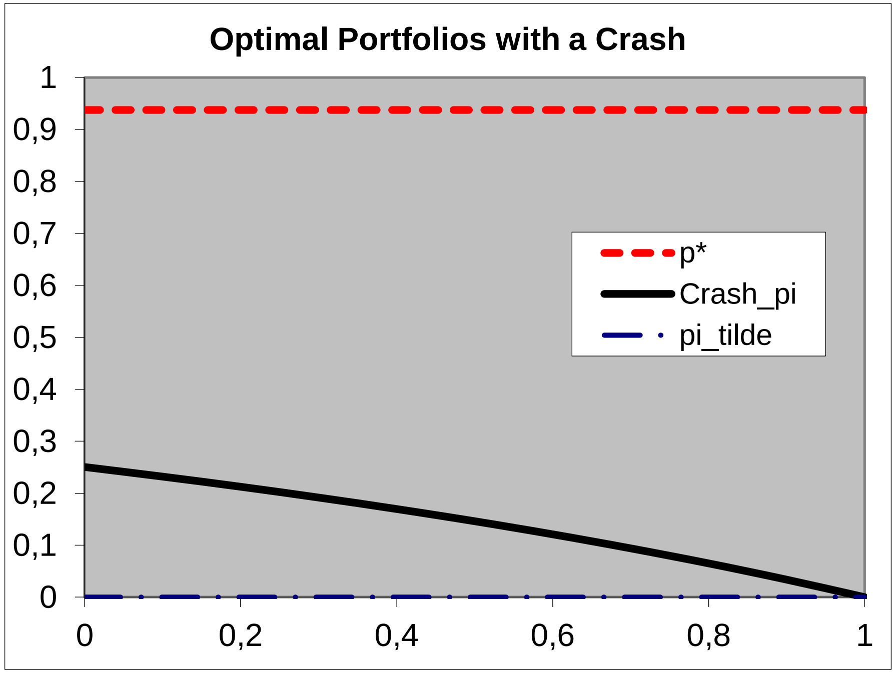
Theorem 1 (K., Menkens (2005)) **"Optimal investment in the face of a crash"**

The unique strategie $\hat{\pi}(.)$ solving (WP) is given as the unique solution $\hat{\pi}(.) \in \left[0, 1/k^*\right)$ of

(2) $$\pi'(t) = -\frac{\gamma\sigma^2}{2k^*}\left(1-\pi(t)k^*\right)\left(\pi(t)-\hat{\pi}(t)\right)^2, \quad \pi(T) = 0$$

before the crash and using the Merton strategy $\hat{\pi}(t)$ after the crash. This strategy makes the investor **<u>indifferent</u>** between the worst crash happening at any (!) time in [0, T] or happening not at all.

Ingredient II: Crash-Modelling and Worst-Case Approach – 5



Data: b = 0.2, r = 0.05, σ = 0.4, k^* = 0,2 und T = 1.

Optimal portfolios with and without crash, π^* = 0,9375.

Main part: The optimal life-time consumption in the face of a crash – 1

Goal: In the crash setting find a pair of portfolio/consumption process $\left(\pi^*(.),c^*(t)\right)$ solving

(WPC) $$\max_{(\pi,c)\in A(x)} \inf_{\tau\in S,0\le k\le k^*} E\left(\int_0^\infty e^{-\delta t}\frac{1}{1-\gamma}\left(c(t)X^{\pi,c}(t)\right)^{1-\gamma}dt\right)$$

with wealth process X(t) and crash time τ and size k given by

(3) $$dX^{\pi,c}(t)=X^{\pi,c}(t)\left[\left(r+\pi(t)\lambda-c(t)\right)dt+\pi(t)\sigma dW(t)\right],X^{\pi,c}(0)=x.$$

(4) $$X^{\pi}(\tau)=X^{\pi}(\tau-)\left(1-\pi(\tau)k\right)$$

Note:

The strategy of the investor consists of a part before the crash $\left(\underline{\pi},\underline{c}\right)$ and a part $\left(\overline{\pi},\overline{c}\right)$ that is used after the crash. We often simplify this by writing $\left(\pi,c\right)$.

Main part: The optimal life-time consumption in the face of a crash – 2

Post crash analysis:

After a crash it is optimal to follow the Merton strategy

(5) $$\hat{\pi}(t) = \frac{\lambda}{\gamma\sigma^2}, \hat{c}(t) = A^{-1/\gamma}, \quad v_0(x) = A\tfrac{1}{1-\gamma}x^{1-\gamma}, \quad A = \left(\tfrac{\delta-(1-\gamma)\psi}{\gamma}\right)^{-\gamma}, \psi = r + \tfrac{1}{2}\frac{\lambda^2}{\gamma\sigma^2}.$$

=> New formulation of (WPC)

(WPC*) $$\max_{(\pi,c)\in A(x)} \inf_{\tau\in S} E\left(\int_0^\tau e^{-\delta t}\tfrac{1}{1-\gamma}\left(c(t)X^{\pi,c}(t)\right)^{1-\gamma}dt + e^{-\delta\tau}v_0\left(\left(1-k*\pi(\tau)\right)X^{\pi,c}(\tau-)\right)\right)$$

Idea: Look at indifference strategies !

(6) $$M^{\pi,c}(s) := \int_0^s e^{-\delta t}\tfrac{1}{1-\gamma}\left(c(t)X^{\pi,c}(t)\right)^{1-\gamma}dt + e^{-\delta s}v_0\left(\left(1-k*\pi(s)\right)X^{\pi,c}(s-)\right)$$

(π, c) is called an indifference strategy if $M^{\pi,c}(t)$ is a martingale

Main part: The optimal life-time consumption in the face of a crash – 3

Proposition 2

For every pair (π, c) with a constant portfolio process $M^{\pi,c}(t)$ is a martingale if we have

(7) $$H(\pi, c) = 0$$

with

(8) $$H(\pi, c) := \tfrac{1}{A}\left(\frac{c}{1-k^*\pi}\right)^{1-\gamma} - \delta + (1-\gamma)\left[\psi - \tfrac{1}{2}\sigma^2(\pi - \hat{\pi})^2 - c\right]$$

Note:

- One can show that for each c > 0 there exists at most one solution

  $\pi = g(c) \in \left[0, \min\{1/k^*, \hat{\pi}\}\right]$ of (7)

- There are infinitely many indifference strategies (π, c)

Challenge:

Is the optimal strategy an indifference strategy? Which indifference strategy is the optimal one?

Main part: The optimal life-time consumption in the face of a crash – 4

An example: $\gamma = 2, \delta = 0.1, k^* = 0.4, \lambda = 0.08, r = 0.05, \sigma = 0.3$

- Merton solution: $\hat{\pi}(t) = 0.44, \hat{c}(t) = 0.084$

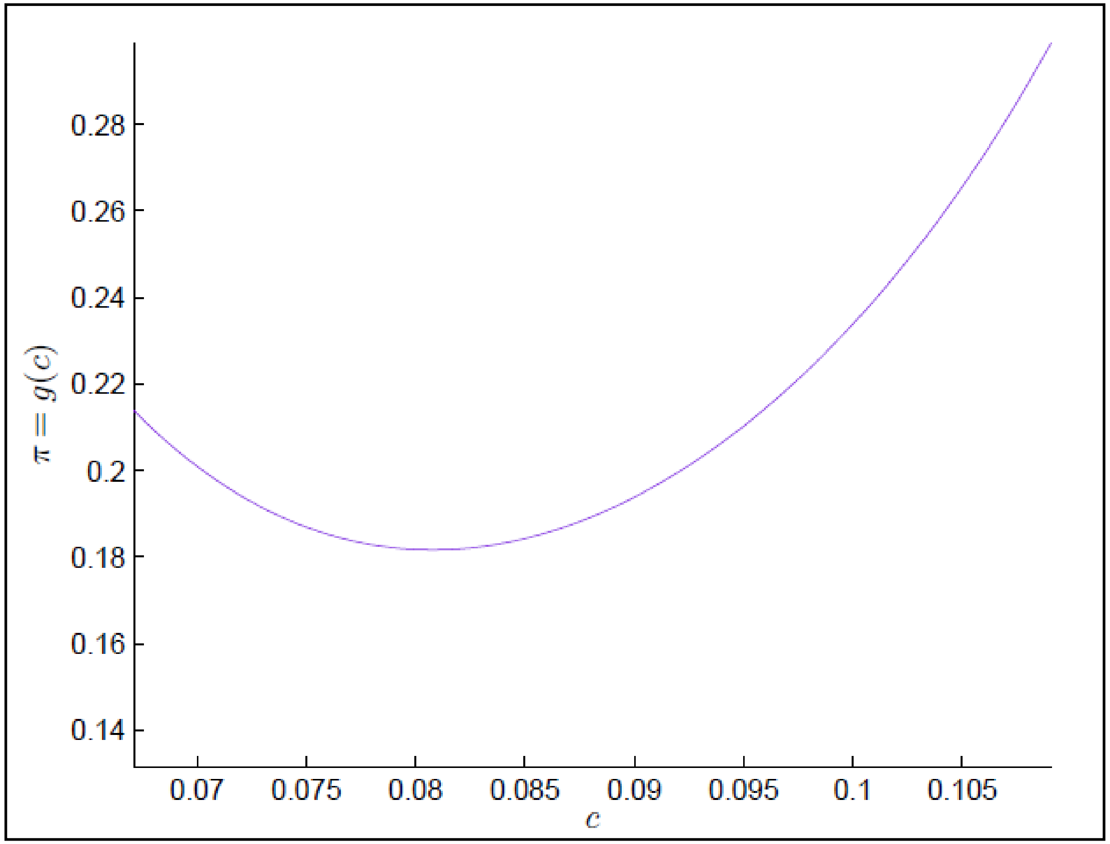


Figure 1: indifference strategies (π, c)

Main part: The optimal life-time consumption in the face of a crash – 5

Main questions:

- Are indifference strategies optimal? => Can be shown similar to Seifried (2010)
- How to determine the optimal indifference strategy?

Two possible approaches:

Idea 1: "Maximization approach"
Find the indifference strategy that performs best if no crash happens:

Theorem 3:
Let $\pi^* \in \left[0, \min\left\{1/k^*, \hat{\pi}\right\}\right]$ denote the unique solution of

(9) $$0 = f(\pi) := \left(1 - k^* x\right)^{1-1/\gamma} - \tfrac{1}{2}\left(1-\gamma\right) A^{1/\gamma} \sigma^2 \left(x - \hat{\pi}\right)^2 - 1$$

and set

(10) $$c^* = A^{-1/\gamma}\left(1 - k^* \pi^*\right)^{1-1/\gamma} = \left(1 - k^* \pi^*\right)^{1-1/\gamma} \hat{c}$$

Then $\left(\pi^*, c^*\right)$ is the unique optimal portfolio-consumption pair before the crash.

Main part: The optimal life-time consumption in the face of a crash – 6

Important (and somewhat surprising!) remark:

1. The investor always follows a smaller portfolio process π^* before the crash than the Merton portfolio, which is optimal after the crash ("**More cautious with respect to stock investment**")
2. **It depends on the investor's character** (i.e. her risk aversion!) if the investor builds up **extra savings for the bad times** to come by consuming less than the Merton investor (in case of $\gamma>1$) or if the investor **consumes more as long as times are good** (in case of $\gamma<1$).

Idea 2: "Minimization approach"

Find the indifference strategy that causes the smallest loss if an immediate crash happens, i.e. determine

(11) $$\pi^* = \min\left\{\pi \,\middle|\, H(\pi, c) = 0 \text{ for some } c \geq 0\right\}$$

Main part: The optimal life-time consumption in the face of a crash – 7

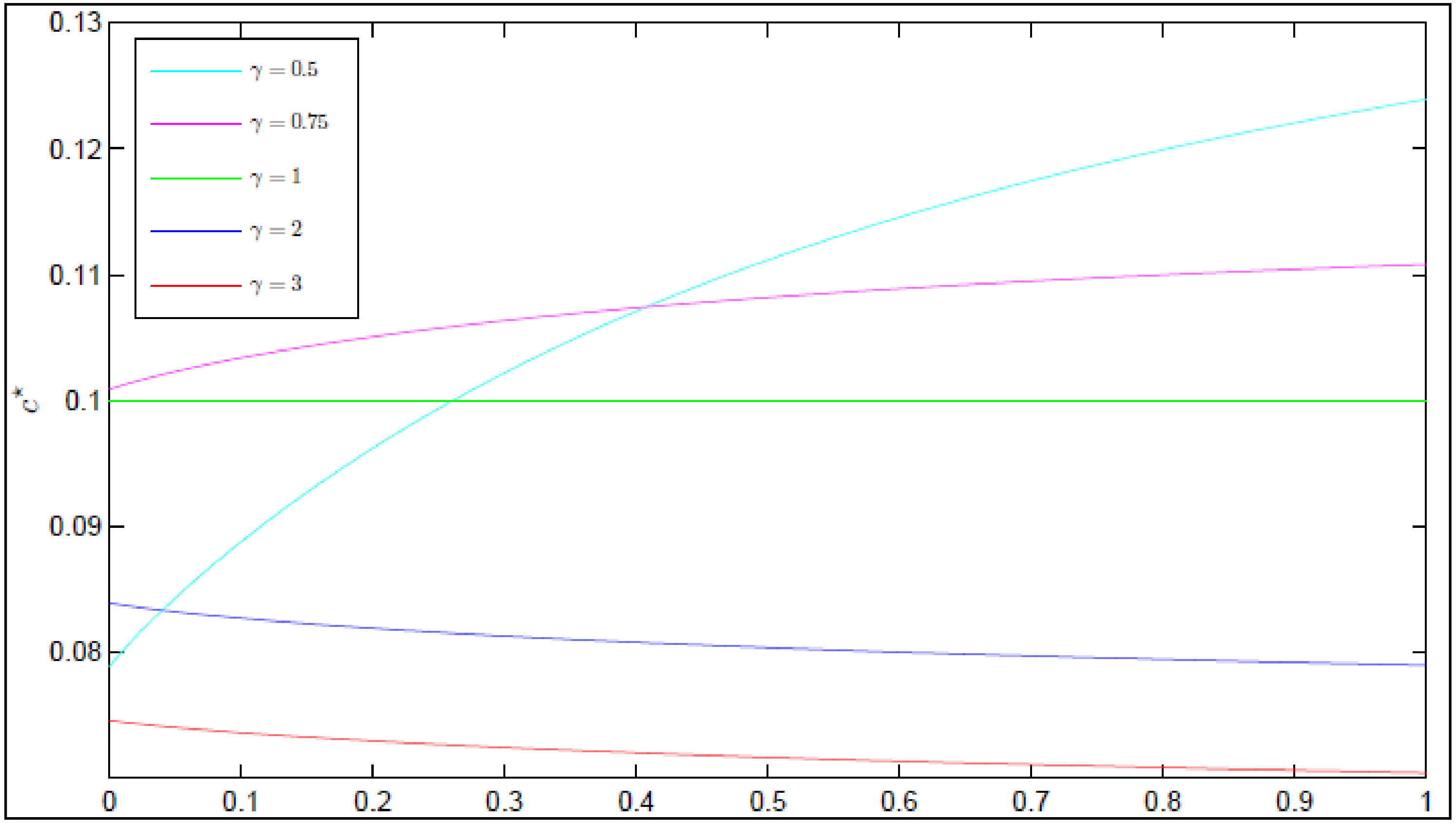


Figure 2: Optimal worst-case consumption as function of the max. crash size

Conclusion: The optimal life-time consumption in the face of a crash

- Everyone tries to protect from a crash via increasing the safe investment
- Depending on the character (=risk aversion) people concentrate on the now or the future after a possible crash

Good or bad ?

Economically this is very reasonable as only this way the economy survives both (!) scenarios!

Thanks for your attention !

References

Davis M.H.A, Norman A.R. (1990) Portfolio selection with transaction costs, *Mathematics of Operations Research* 15, 676–713.

Foldes L. (1990) Conditions for optimality in the infinite horizon portfolio-cum-saving problem with semimartingale investments. *Stochastics and Stochastics Reports* 29, 133–170.

Desmettre S., Korn R., Seifried F.T. (2013) Worst-case consumption-portfolio optimization. Working paper.

Hua P., Wilmott P. (1997) Crash courses, *Risk* 10, 64–67.

Korn R. (1998) Portfolio optimisation with strictly positive transaction costs and impulse control. Finance and Stochastics 2, 85–114.

Korn R., Menkens O. (2005) Worst-case scenario portfolio optimization: A new stochastic control approach. *Mathematical Methods of Operations Research* 62, 123–140.

Korn R., Steffensen M. (2007) On worst-case portfolio optimization. *SIAM Journal on Control and Optimization* 46, 2013–2030.

Korn R., Wilmott (2002) Optimal portfolios under the threat of a crash. *International Journal of Theoretical and Applied Finance* 5, 171–187.

Kramkov D., Schachermayer W. (1999) The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability* 9, 904–950.

Merton R.C. (1969) Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and Statistics* 51, 247–257.

Mossin J. (1968) Optimal multiperiod portfolio policies. *Journal of Business* 41, 215–229.

Samuelson P. (1969) Lifetime portfolio selection by dynamic stochastic programming. *Review of Economics and Statistics* 51, 239–246.

Seifried F.T. (2010) Optimal investment for worst-case crash scenarios: A martingale approach. *Mathematics of Operations Research* 35, 559–579.