Save for Bad Times
or Consume as Long as You Have?

Worst-Case Optimal Life-Time Consumption

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A preliminary remark:

- Assume that possibly during your life-time you may experience a once-in-a-life-time catastrophe (World war, Great Depression, Subprime Crisis, ...)

- Assume that there is no probability to judge how likely it is to experience it (“Knightian Uncertainty”)

- Assume that at that event you lose an a priori unknown fraction of your wealth invested in stocks

  Would you build up extra savings for the bad times

  or

  would you spend more as long as you have it?
Ingredient I: Optimal Life-Time Consumption – 1

Problem:
How to distribute your consumption (as a fraction of your wealth) over your life-time?

References: Optimal Life-Time Consumption Problems are treated in
- Mossin (1968), Samuelson (1969) – Discrete-time model
- Merton (1969) – Continuous-time model
- Foldes (1990), Kramkov, Schachermayer (1999) – General utility function and semimartingale prices
- Davis, Norman (1990) – Continuous-time with proportional transaction costs
- Korn (1998) – Continuous-time with general transaction costs
- + many more
Ingredient I: Optimal Life-Time Consumption – 2

The Merton problem

In a Black-Scholes-type market with a money market account and a stock given by

\[ dP_0(t) = P_0(t) r dt, \quad P_0(0) = 1, \quad \text{“MMA”} \]

\[ dP_1(t) = P_1(t)((r + \lambda) dt + \sigma dW(t)), \quad i = 1, \ldots, n, \quad P_1(0) = p_1, \quad \text{“Stock”}, \]

**Goal:** Find a pair of portfolio/consumption process \((\hat{\pi}(\cdot), \hat{c}(t))\) solving

\[
\max_{(\pi, c) \in A(x)} \mathbb{E} \left( \int_0^\infty e^{-\delta t} \frac{1}{1-\gamma} \left( c(t) X_{\pi,c}(t) \right)^{1-\gamma} dt \right)
\]

with wealth process

\[
dX_{\pi,c}(t) = X_{\pi,c}(t) \left[ (r + \pi(t) \lambda - c(t)) dt + \pi(t) \sigma dW(t) \right], \quad X_{\pi,c}(0) = x.
\]

**The Merton solution (Merton (1969)):**

\[
\hat{\pi}(t) = \frac{\lambda}{\gamma \sigma^2}, \quad \hat{c}(t) = A^{-1/\gamma}, \quad v_0(x) = A^{-\frac{1-\gamma}{1-\gamma}} x^{1-\gamma}, \quad A = \left( \frac{\delta-(1-\gamma)\psi}{\gamma} \right)^{-\gamma}, \quad \psi = r + \frac{1}{2} \frac{\lambda^2}{\gamma \sigma^2}
\]

where

\[
v_0(x) := \sup_{(\pi, c) \in A(x)} \mathbb{E}^x \left( \int_0^\infty e^{-\delta t} \frac{1}{1-\gamma} \left( c(t) X_{\pi,c}(t) \right)^{1-\gamma} dt \right)
\]

is the value function.
**Ingredient II: Crash-Modelling and Worst-Case Approach – 1**


1. Hua & Wilmott (1997): “Number and size of crashes until a given time horizon are bounded” ⇒ no probabilistic assumptions on height, number and time of crashes.


**For simplicity:** At most one crash in \([0, T]\) with maximum height \(k^* < 1\).

\[
dP_0(t) = P_0(t) r \, dt, \quad P_0(0) = 1, \quad \text{“MMA”}
\]

\[
dP_1(t) = P_1(t) \left( (r + \lambda) \, dt + \sigma \, dW(t) \right), \quad P_1(0) = p, \quad \text{“Stock in normal times”}
\]

**At crash time:** Stock price falls by a fraction \(k \in [0, k^*] \)

**Consequence:** Wealth process \(X^\pi(t)\) at crash time:

\[
X^\pi(t^-) = \left( 1 - \pi(t) \right) X^\pi(t^-) + \pi(t) X^\pi(t^-) \\
\Rightarrow \left( 1 - \pi(t) \right) X^\pi(t^-) + \pi(t) X^\pi(t^-)(1-k) = X^\pi(t^-)(1-\pi(t)k) = X^\pi(t)
\]
**Ingredient II: Crash-Modelling and Worst-Case Approach – 2**

**Aim:** Find the best uniform **worst-case bound**, i.e. solve

$$(WP) \sup_{\pi(.) \in A(x)} \inf_{0 \leq t \leq T, 0 \leq k \leq k^*} E\left(\frac{1}{1-\gamma}\left(X^\pi(T)\right)^{1-\gamma}\right)$$

with final wealth $X^\pi(T) = (1 - \pi(t)k)\tilde{X}^\pi(T)$ in case of a crash of size $k$ at time $t$. 


**Ingredient II: Crash-Modelling and Worst-Case Approach – 3**

Two extreme strategies (for log-utility, i.e. $\gamma=0$):

i) $\pi(t) \equiv 0$: “Play safe”

$\Rightarrow$ Worst-case szenario: no crash (!) with worst-case bound

$$WCB_0 = E\left(ln\left(X^0(T)\right)\right) = ln(x) + rT$$

ii) $\pi(t) \equiv \pi^* := \frac{b-r}{\sigma^2}$: “Optimal investment in the crash-free world ”

$\Rightarrow$ worst-case szenario: crash of maximum height $k^*$ (at any time !) with worst-case bound

$$WCB_{\pi^*} = E\left(ln\left(X^{\pi^*}(T)\right)\right) = ln(x) + rT + \frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 T + ln\left(1 - \pi^* k^*\right)$$

**Insights:**

- it depends on $T$ which strategy performs better
- a constant portfolio **cannot** be optimal
- Strategy i) is too conservative to be good if no crash happens, strategie ii) is too risky to perform well if a high crash appears

$\Rightarrow$ **an optimal strategy should balance this out**!
**Ingredient II: Crash-Modelling and Worst-Case Approach – 4**

**Theorem 1** (K., Menkens (2005)) *“Optimal investment in the face of a crash”*

The unique strategy \( \hat{\pi}(.) \) solving (WP) is given as the unique solution \( \hat{\pi}(.) \in \left[ 0, \frac{1}{k^*} \right) \) of

\[
\pi'(t) = -\frac{\gamma \sigma^2}{2k^*} \left( 1 - \pi(t)k^* \right) \left( \pi(t) - \hat{\pi}(t) \right)^2,
\]

before the crash and using the Merton strategy \( \hat{\pi}(t) \) after the crash. This strategy makes the investor **indifferent** between the worst crash happening at any (!) time in \([0, T]\) or happening not at all.
Ingredient II: Crash-Modelling and Worst-Case Approach – 5

Data: $b = 0.2, r = 0.05, \sigma = 0.4, k^* = 0.2$ und $T = 1$.

Optimal portfolios with and without crash, $\pi^* = 0.9375$. 
**Main part: The optimal life-time consumption in the face of a crash – 1**

**Goal:** In the crash setting find a pair of portfolio/consumption process \((\pi^*(\cdot), c^*(t))\) solving

\[
(WPC) \quad \max_{(\pi,c)\in A(x)} \inf_{\tau\in S, 0 \leq k \leq k^*} E \left( \int_0^\infty e^{-\delta t} \frac{1}{1-\gamma} \left( c(t) X^{\pi,c}(t) \right)^{1-\gamma} dt \right)
\]

with wealth process \(X(t)\) and crash time \(\tau\) and size \(k\) given by

\[
(3) \quad dX^{\pi,c}(t) = X^{\pi,c}(t) \left[ (r + \pi(t) \lambda - c(t)) dt + \pi(t) \sigma dW(t) \right], \quad X^{\pi,c}(0) = x.
\]

\[
(4) \quad X^\pi(\tau) = X^\pi(\tau-)(1 - \pi(\tau)k)
\]

**Note:**

The strategy of the investor consists of a part before the crash \((\pi, c)\) and a part \((\overline{\pi}, \overline{c})\) that is used after the crash. We often simplify this by writing \((\pi, c)\).
Main part: The optimal life-time consumption in the face of a crash – 2

Post crash analysis:

After a crash it is optimal to follow the Merton strategy

\( \hat{\pi}(t) = \frac{\lambda}{\gamma \sigma^2}, \hat{c}(t) = A^{1/\gamma}, \quad v_0(x) = A \frac{1}{1-\gamma} x^{1-\gamma}, \quad A = \left( \frac{\delta - (1-\gamma)\psi}{\gamma} \right)^{-\gamma}, \quad \psi = r + \frac{1}{2} \frac{\lambda^2}{\gamma \sigma^2}. \)

=> New formulation of (WPC)

\[
(WPC^*) \quad \max_{(\pi, c) \in \mathcal{A}(x)} \inf_{\tau \in \mathcal{S}} E \left( \int_0^\tau e^{-\delta t} \frac{1}{1-\gamma} \left( c(t) X^{\pi, c}(t) \right)^{1-\gamma} dt + e^{-\delta \tau} v_0 \left( (1 - k * \pi(\tau)) X^{\pi, c}(\tau -) \right) \right)
\]

Idea: Look at indifference strategies!

\( M^{\pi, c}(s) := \int_0^s e^{-\delta t} \frac{1}{1-\gamma} \left( c(t) X^{\pi, c}(t) \right)^{1-\gamma} dt + e^{-\delta s} v_0 \left( (1 - k * \pi(s)) X^{\pi, c}(s -) \right) \)

\((\pi, c)\) is called an indifference strategy if \( M^{\pi, c}(t) \) is a martingale
Main part: The optimal life-time consumption in the face of a crash – 3

Proposition 2
For every pair \((\pi, c)\) with a constant portfolio process \(M^{\pi,c}(t)\) is a martingale if we have

\[ H(\pi, c) = 0 \]

with

\[ H(\pi, c) := \frac{1}{A}\left(\frac{c}{1 - k^*\pi}\right)^{1-\gamma} - \delta + (1 - \gamma)\left[\psi - \frac{1}{2}\sigma^2(\pi - \hat{\pi})^2 - c\right] \]

Note:
- One can show that for each \(c > 0\) there exists at most one solution
  \[ \pi = g(c) \in \left[0, \min \{1/k^*, \hat{\pi}\}\right] \text{ of } (7) \]
- There are infinitely many indifference strategies \((\pi, c)\)

Challenge:
Is the optimal strategy an indifference strategy? Which indifference strategy is the optimal one?
Main part: The optimal life-time consumption in the face of a crash – 4

An example: $\gamma = 2, \delta = 0.1, k^* = 0.4, \lambda = 0.08, r = 0.05, \sigma = 0.3$

- Merton solution: $\hat{\pi}(t) = 0.44, \hat{c}(t) = 0.084$

Figure 1: indifference strategies $(\pi, c)$
Main part: The optimal life-time consumption in the face of a crash – 5

Main questions:

- Are indifference strategies optimal? => Can be shown similar to Seifried (2010)
- How to determine the optimal indifference strategy?

Two possible approaches:

**Idea 1: “Maximization approach”**
Find the indifference strategy that performs best if no crash happens:

**Theorem 3:**
Let \( \pi^* \in \left[ 0, \min\{1/k^*, \hat{\pi}\} \right] \) denote the unique solution of

\[
0 = f(\pi) := (1 - k^* x)^{1-1/\gamma} - \frac{1}{2} (1 - \gamma) A^{1/\gamma} \sigma^2 (x - \hat{\pi})^2 - 1
\]

and set

\[
c^* = A^{-1/\gamma} (1 - k^* \pi^*)^{1-1/\gamma} = (1 - k^* \pi^*)^{1-1/\gamma} \hat{c}
\]

Then \((\pi^*, c^*)\) is the unique optimal portfolio-consumption pair before the crash.
Main part: The optimal life-time consumption in the face of a crash – 6

Important (and somewhat surprising!) remark:

1. The investor always follows a smaller portfolio process \( \pi^* \) before the crash than the Merton portfolio, which is optimal after the crash ("More cautious with respect to stock investment")

2. It depends on the investor’s character (i.e. her risk aversion!) if the investor builds up extra savings for the bad times to come by consuming less than the Merton investor (in case of \( \gamma > 1 \)) or if the investor consumes more as long as times are good (in case of \( \gamma < 1 \)).

Idea 2: “Minimization approach”

Find the indifference strategy that causes the smallest loss if an immediate crash happens, i.e. determine

\[
\pi^* = \min \{ \pi \mid H (\pi, c) = 0 \text{ for some } c \geq 0 \}
\]
Main part: The optimal life-time consumption in the face of a crash – 7

Figure 2: Optimal worst-case consumption as function of the max. crash size
Conclusion: The optimal life-time consumption in the face of a crash

- Everyone tries to protect from a crash via increasing the safe investment
- Depending on the character (=risk aversion) people concentrate on the now or the future after a possible crash

Good or bad?

Economically this is very reasonable as only this way the economy survives both (!) scenarios!

Thanks for your attention!


