Two Price Valuation with Applications to Actuarial Problems

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Law of one Price

In complete markets and for liquid assets

\[ E^Q[X] \]

Reality however is incomplete and liquidity might be poor

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bid price  |  ask price
(quick seller)  |  (quick buyer)
Example for a Two Price Evaluation

Consider a public debt obligation (e.g. Greek government bond)

Possibility of default or other causes of illiquidity:
  The lender (buyer) has to discount the value of the loan or bond
  $\rightarrow$ bid price

The borrower (issuer of debt) does not contemplate default:
  For him the obligation is risk free $\rightarrow$ ask price

At maturity: convergence of bid and ask

Consequences:  
  • Bid price will vary with the changes in the borrower’s credit status
  • Ask price will remain relatively steady

Further fact: Opponent of trade depends on the direction of the trade
Acceptability of Cashflows

Outcome (cashflow) of a risky position: $X$ random variable

In complete and perfectly liquid markets: unique pricing kernel given by a probability measure $Q$

value of the position: $E^Q[X]$

position is acceptable if: $E^Q[X] \geq 0$

company’s objective is: maximize $E^Q[X]$

Real markets:

Instead of a unique probability measure $Q$ we have to consider a set of probability measures (scenarios) $Q \in \mathcal{M}$

$$E^Q[X] \geq 0 \quad \text{for all } Q \in \mathcal{M} \quad \text{or} \quad \inf_{Q \in \mathcal{M}} E^Q[X] \geq 0$$
Coherent Risk Measures

Specification of $\mathcal{M}$ (test measures, generalized scenarios)

Axiomatic theory of risk measures: desirable properties

Monotonicity: $X \geq Y \implies \varphi(X) \leq \varphi(Y)$

Cash invariance: $\varphi(X + c) = \varphi(X) - c$

Scale invariance: $\varphi(\lambda X) = \lambda \varphi(X), \quad \lambda \geq 0$

Subadditivity: $\varphi(X + Y) \leq \varphi(X) + \varphi(Y)$

Examples: Value at Risk (VaR)

Tail-VaR (expected shortfall)

General risk measure: $\varphi_m(X) = -\int_0^1 q_u(X)m(du)$

Any coherent risk measure has a representation

$\varphi(X) = -\inf_{Q \in \mathcal{M}} E^Q[X]$
Operationalization

Link between acceptability and concave distortions
(Cherny and Madan (2009))

→ Concave distortions

Assume acceptability is completely defined by the distribution function of the risk

\[ \Psi(u) : \text{concave distribution function on } [0, 1] \]

⇒ \[ M \] the set of supporting measures is given by all measures \( Q \) with density \( Z = \frac{dQ}{dP} \) s.t.

\[ E^P[(Z - a)^+] \leq \sup_{u \in [0,1]} (\Psi(u) - ua) \quad \text{for all } a \geq 0 \]

Acceptability of \( X \) with distribution function \( F(x) \)

\[ \int_{-\infty}^{+\infty} xd\Psi(F(x)) \geq 0 \]
Distortion

\[ \Psi(x) \]

\[ \gamma = 2 \quad \gamma = 10 \quad \gamma = 20 \quad \gamma = 100 \]
Families of Distortions (1)

Consider families of distortions \( (\psi^\gamma)_{\gamma \geq 0} \)

\( \gamma \) stress level

Example: MIN VaR

\[
\psi^\gamma(x) = 1 - (1 - x)^{1+\gamma} \quad (0 \leq x \leq 1, \gamma \geq 0)
\]

Statistical interpretation:

Let \( \gamma \) be an integer, then \( \varrho_\gamma(X) = -E(Y) \) where

\[
Y \overset{\text{law}}{=} \min\{X_1, \ldots, X_{\gamma+1}\}
\]

and \( X_1, \ldots, X_{\gamma+1} \) are independent draws of \( X \)
Families of Distortions (2)

Further examples: MAX VaR

\[ \Psi^\gamma(x) = x^{1/(1+\gamma)} \quad (0 \leq x \leq 1, \gamma \geq 0) \]

Statistical interpretation: \( \varrho_\gamma(X) = -E[Y] \)

where \( Y \) is a random variable s.t.

\[ \max\{Y_1, \ldots, Y_{\gamma+1}\} \text{ law} = X \]

and \( Y_1, \ldots, Y_{\gamma+1} \) are independent draws of \( Y \).

Combining MIN VaR and MAX VaR: MAX MIN VaR

\[ \Psi^\gamma(x) = (1 - (1 - x)^{1+\gamma})^{1/(1+\gamma)} \quad (0 \leq x \leq 1, \gamma \geq 0) \]

Interpretation: \( \varrho_\gamma(X) = -E[Y] \) with \( Y \) s.t.

\[ \max\{Y_1, \ldots, Y_{\gamma+1}\} \text{ law} = \min\{X_1, \ldots, X_{\gamma+1}\} \]
Families of Distortions (3)

Distortion used: MIN MAX VaR

\[ \Psi^\gamma(x) = 1 - \left(1 - x^{\frac{1}{1+\gamma}}\right)^{1+\gamma} \quad (0 \leq x \leq 1, \gamma \geq 0) \]

\[ \varrho^\gamma(X) = -E[Y] \quad \text{with } Y \text{ s.t.} \quad Y \overset{\text{law}}{=} \min\{Z_1, \ldots, Z_{\gamma+1}\}, \]
\[ \max\{Z_1, \ldots, Z_{\gamma+1}\} \overset{\text{law}}{=} X \]
Families of Distortions (4)

\[ \psi(x) \]

- $\gamma = 0.50$
- $\gamma = 0.75$
- $\gamma = 1.0$
- $\gamma = 5.0$

\[ \psi(x) = \begin{cases} 
\frac{1}{\gamma} & \text{for } x \leq \frac{1}{\gamma} \\
\frac{\gamma - 1}{\gamma} x & \text{for } x > \frac{1}{\gamma} 
\end{cases} \]

Introduction
- Acceptability
- Bid and Ask
- Continuous Time
- Perpetuities
- Insurance Losses
- References
Marking Assets and Liabilities

Assets: Cash flow to be received \( X \geq 0 \)

Largest value \( b(X) \) s.t. \( X - b(X) \) is acceptable

\[ b(X) = \inf_{Q \in \mathcal{M}} E^Q[X] \]

Bid price or Lower price

Liabilities: Cash flow to be paid out \( X \geq 0 \)

Smallest value \( a(X) \) s.t. \( a(X) - X \) is acceptable

\[ a(X) = \sup_{Q \in \mathcal{M}} E^Q[X] \]

Ask price or Upper price
Directional Prices in a Two Price Economy

The goal is not to get a single risk neutral price which could be interpreted as a midpoint between bid and ask.

Instead modeling two separate prices at which transactions occur

\[\rightarrow\] directional prices

**Bid price:** Minimal conservative valuation s.t. the expected outcome will safely exceed this price

**Ask price:** Maximal valuation s.t. the expected payout will fall below this price

\[\rightarrow\] specification of the set of valuation possibilities
Relating Bid and Ask Prices

Consider real-valued cashflows $X$, e.g. swaps

$$X = X^+ - X^-$$

$$\Rightarrow b(X) = b(X^+) - a(X^-)$$

and $$a(X) = a(X^+) - b(X^-)$$

Valuation as asset: $X^+$ is an asset and priced at the bid
$X^-$ is a liability and priced at the ask

Valuation as liability: $X^-$ is an asset and priced at the bid
$X^+$ is a liability and priced at the ask
Explicit Bid and Ask Pricing

Bid Price of a cash flow $X$: Acceptability of $X - b(X)$

$$b(X) = \int_{-\infty}^{\infty} xd\psi(F_X(x))$$

Ask Price of a cash flow $X$: Acceptability of $a(X) - X$

$$a(X) = -\int_{-\infty}^{\infty} xd\psi(1 - F_X(-x))$$

Examples: Calls and Puts

$$bC(K, t) = \int_{K}^{\infty} (1 - \psi(F_{S_t}(x))) \, dx$$

$$aC(K, t) = \int_{K}^{\infty} \psi(1 - F_{S_t}(x)) \, dx$$

$$bP(K, t) = \int_{0}^{K} (1 - \psi(1 - F_{S_t}(x))) \, dx$$

$$aP(K, t) = \int_{0}^{K} \psi(F_{S_t}(x)) \, dx$$
Continuous Time Theory

Underlying uncertainty given by a pure jump Lévy process \((X_t)_{0 \leq t \leq T}\)

Specified by: drift term \(\alpha\), Lévy measure \(k(y)dy\) \((y \neq 0)\)

Example: Variance gamma

\[
k(y) = \frac{C}{|y|} (\exp(-G|y|)1_{\{y<0\}} + \exp(-M|y|)1_{\{y>0\}})
\]

Note that

\[
\int_{\mathbb{R}} y^2 k(y)dy < \infty
\]

Infinitesimal generator \(\mathcal{L}\) of the process

\[
\mathcal{L}u(x) = \alpha \frac{\partial u}{\partial x}(x) + \int_{\mathbb{R}} \left( u(x + y) - u(x) - \frac{\partial u}{\partial x}(x)y \right) k(y)dy
\]
GH Levy process with marginal densities

values of GH (-0.5,100,0,1,0.1) Levy process

0.0 0.5 1.0 1.5 2.0
99.8 100.0 100.2 100.4 100.6

t
Variance gamma density

\[ C = 5 \\
G = 1 \\
M = 10 \]
Valuation of Financial Contracts

Consider a claim which pays $\phi(X_t)$ at time $t$

Denote by $u(x, t)$ its time zero value when $X_0 = x$

\[ \Rightarrow u(x, t) = E \left[ e^{-rt} \phi(X_t) \mid X_0 = x \right] \]

risk-neutral value
for constant interest rate $r$

$u(x, t)$ is at the same time the solution of the partial integro-differential equation (PIDE)

\[ u_t = \mathcal{L}(u) - ru \]

with boundary condition $u(x, 0) = \phi(x)$
\( \mathcal{G} \)-Expectations Using Distortions (1)

Remember: Higher weight on unfavorable states
Lower weight on favorable states

Integral part of the PIDE

\[
\int_{\mathbb{R}} \left( u(x + y, t) - u(x, t) - u_x(x, t)y \right) \frac{y^2 k(y) dy}{y^2} \cdot g(y) dy =: Y_{x, t}
\]

where \( g(y) = \frac{y^2 k(y)}{\int_{\mathbb{R}} y^2 k(y) dy} \)

\( \rightarrow \) \( g \) is a probability density

Define the distribution function

\[
F_{Y_{x, t}}(v) = \int_{A(x, t, v)} g(y) dy
\]

where \( A(x, t, v) = \{y \mid Y_{x, t} \leq v\} \)
\( G \)-Expectations Using Distortions (2)

Integral part of the PIDE is now

\[
\int_{\mathbb{R}} v dF_{Y_{x,t}}(v)
\]

Distorted expectation

\[
\int_{\mathbb{R}} v d\Psi(F_{Y_{x,t}}(v))
\]

which by decomposition can be written as

\[
- \int_{-\infty}^{0} \Psi(P^g(Y_{x,t} \leq v)) dv + \int_{0}^{\infty} (1 - \Psi(P^g(Y_{x,t} \leq v))) dv
\]

Define the new (distorted) operator

\[
G_{QV} u(x) = \alpha \frac{\partial u}{\partial x}(x) - \int_{-\infty}^{0} \Psi(P^g(Y_{x,t} \leq v)) dv + \int_{0}^{\infty} (1 - \Psi(P^g(Y_{x,t} \leq v)))) dv
\]

and solve the (distorted) PIDE

\[
u_t = G_{QV}(u) - ru
\]
Alternative $\mathcal{G}$-Expectation Approach

Truncation of the Lévy measure

$$
\int_{\{|y| \geq \epsilon\}} (u(x + y, t) - u(x, t) - u_x(x, t)y) k(y) dy
$$

Definition of a probability density $h(y)$ via

$$
\int_{\{|y| \geq \epsilon\}} (u(x + y, t) - u(x, t) - u_x(x, t)y) \left( \int_{\{|y| \geq \epsilon\}} k(y) dy \right) h(y) dy
$$

$$
=: \tilde{Y}_{x,t}
$$

where $h(y) = \frac{k(y)}{\int_{\{|y| \geq \epsilon\}} k(y) dy} 1_{\{|y| \geq \epsilon\}}$

The distorted operator is now

$$
\mathcal{G}_{NL} u(x) = \alpha \frac{\partial u}{\partial x} (x) - \int_0^\infty \psi(P^h(\tilde{Y}_{x,t} \leq v)) dv + \int_0^\infty (1 - \psi(P^h(\tilde{Y}_{x,t} \leq v))) dv
$$
Perpetuities

The discounted variance gamma model
\( \gamma_p(t), \gamma_n(t) \) two independent standard gamma processes

Driving process

\[
X(t) = \int_0^t b_p e^{-rs} d\gamma_p(c_p s) - \int_0^t b_n e^{-rs} d\gamma_n(c_n s)
\]

\( b_p, c_p, b_n, c_n > 0 \) scale and shape parameters of the undiscounted gamma processes

Discounted stock price

\[
M(t) = \exp(X(t) + \omega(t))
\]

where \( \exp(\omega(t)) = (E[\exp(X(t))])^{-1} \)

\( \rightarrow \) uniformly integrable martingale with a well-defined limit

\[
M(\infty) = \exp(X(\infty) + \omega(\infty))
\]
Valuation of Perpetuities

Consider now a claim promising at infinity $F(M(\infty))$ where the payout is expressed in time zero dollars ($F$ ‘nice’ function)

Value of the claim at time $t$

$$w_F(t) = E[F(M(\infty)) | \mathcal{F}_t]$$

→ martingale

Observe now that

$$X(\infty) = X(t) + \int_t^\infty b_p e^{-ru} d\gamma_p(c_p u) - \int_t^\infty b_n e^{-ru} d\gamma_n(c_n u)$$

$$(d) \equiv X(t) + e^{-rt} Y$$

for an independent random variable $Y \sim X(\infty)$

$\Rightarrow$

$$w_F(t) = H(X(t), e^{-rt})$$
Bid and Ask Prices (1)

Martingale condition on \( w_F(t) \) (write \( v = e^{-rt} \))

\[
-rvH_v + \int_{-\infty}^{-\infty} (H(X + y, v) - H(X, v))k(y, v)dy = 0
\]

PIDE with boundary condition

\[
H(X, 0) = F(\exp(X(\infty) + \omega(\infty)))
\]

where

\[
k(y, v) = \frac{c_p}{y} \exp \left( - \frac{y}{b_p v} \right) 1_{\{y > 0\}} + \frac{c_n}{|y|} \exp \left( - \frac{|y|}{b_n v} \right) 1_{\{y < 0\}}
\]
Bid and Ask Prices (2)

Rewrite the PIDE

\[ rvH_v = \int_{-\infty}^{+\infty} \frac{(H(X + y, \nu) - H(X, \nu)) \int_{-\infty}^{+\infty} y^2 k(y, \nu) dy}{y^2} dF_{QV}(y) \]

where

\[ F_{QV}(a) = \frac{1}{\int_{-\infty}^{+\infty} y^2 k(y, \nu) dy \int_{-\infty}^{a} y^2 k(y, \nu) dy} \]

Bid price is the solution of the distorted PIDE

\[ rvH_v = \int_{-\infty}^{+\infty} \frac{(H(X + y, \nu) - H(X, \nu)) \int_{-\infty}^{+\infty} y^2 k(y, \nu) dy}{y^2} d\Psi^\gamma(F_{QV}(y)) \]

Ask price: Negative of the bid price of the negative cash flow
Implementation Details

Risk neutral parameters from S & P 500

\[ r = 0.02966 \quad b_p = 0.0145 \quad c_p = 48.4215 \]

\[ b_n = 0.5707 \quad c_n = 0.3493 \]

Actually solved for a PIDE in \( M(t) \):

\[ G(M(v), v) = M(v) \exp \left( \omega(\infty) - \omega\left( -\frac{\ln v}{r} \right) \right) \phi_Y(-iv) \]
Bid and Ask as a function of Time for 3 spot levels
Two Price Valuation of Insurance Losses (1)

Cumulated loss process \( L(t) \): e.g. compound Poisson (arrival rate \( \lambda \))
Loss sizes are iid \( \gamma \)-distributed (scale and shape parameters \( \zeta \) and \( \kappa \))
Consider the value process in time zero dollars

\[
V(t) = E_t \left[ \int_0^\infty e^{-rs} dL(s) \right]
\]

Let \( X(t) \) be the discounted losses to date

\[
X(t) = \int_0^t e^{-rs} dL(s)
\]

Rewrite

\[
\int_0^\infty e^{-rs} dL(s) = X(t) + e^{-rt} \int_t^\infty e^{-r(s-t)} dL(s) \stackrel{(d)}{=} X(t) + e^{-rt} Y
\]

where \( Y \) is an independent copy of \( \int_0^\infty e^{-rs} dL(s) \)

\[
\Rightarrow \quad V(t) = H(X(t), e^{-rt})
\]
Two Price Valuation of Insurance Losses (2)

Applying Itô’s formula and using the martingale condition (where we replace \( t \) by \( v = e^{-rt} \))

\[
rvH_v = \int_0^\infty (H(x + w, v) - H(X, v))k(w, v)dw
\]

where \( k(w, v) \) is related to the Lévy system for \( X(t) \)

\[
k(w, v) = \frac{\lambda}{\Gamma(\kappa)} \left( \frac{\zeta}{v} \right)^\kappa w^{\kappa - 1} \exp \left( - \frac{\zeta}{v} w \right)
\]

Risk neutral price is the solution of this PIDE

Bid price is the solution of the distorted PIDE

How to distort a measure integral?
Measure Distortions (1)

Consider a possibly infinite measure $\mu$ with tails of finite measure and

$$m = \int_{-\infty}^{+\infty} \nu(y) \mu(dy) < \infty$$

Rewrite this as

$$m = -\int_{-\infty}^{0} \mu(\nu(y) \leq x) \, dx + \int_{0}^{\infty} \mu(\nu(y) > x) \, dx$$

Distorted measure integrals

$$m = -\int_{-\infty}^{0} \Gamma_+(\mu(\nu(y) \leq x)) \, dx + \int_{0}^{\infty} \Gamma_-(\mu(\nu(y) > x)) \, dx$$

for functions $\Gamma_+, \Gamma_- : \mathbb{R}_+ \to \mathbb{R}_+$

$\Gamma_{\pm}(0) = 0$, monotone increasing, resp. concave and convex, bounded below and above by the identity function
Measure Distortions (2)

Natural Candidates:

\[ \Gamma_+(x) = x + \alpha (1 - e^{-cx})^{-\frac{1}{1+\gamma_+}} \]

\[ \Gamma_-(x) = x - \frac{\beta}{c} (1 - e^{-c(1+\gamma_-)x}) \]

(\(\Gamma_+\) derived from maxvar, \(\Gamma_-\) from minvar)
Bid Price for the Discounted Cumulated Loss Process

Distorted measure integral for positive variables

\[ m = \int_{0}^{\infty} \Gamma_{-}(\mu(\chi > x)) dx \]

Rewrite this as (integration by parts)

\[ m = -\int_{0}^{\infty} xd\Gamma_{-}(\mu(\chi > x)) \]

Now choose \( \chi(y) = H(X + y, \nu) - H(X, \nu), \quad \mu(dy) = k(y, \nu) dy \)

Bid price is the solution of

\[ rvH_{\nu} = -\int_{0}^{\infty} xd\Gamma_{-}(\mu(\chi > x)) \]

For the ask price one has to consider the integral

\[ rvH_{\nu} = -\int_{0}^{\infty} xd\Gamma_{+}(\mu(\chi > x)) \]


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