Convex order approximations in case of cash flows of mixed signs

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Outline

1 Motivation

2 Problem description

3 Saving & Terminal Wealth

4 Reserves for Future Obligations

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J. Dhaene, M. Goovaerts, M. Vanmaele, and K. Van Weert.
Convex order approximations in case of cash flows of mixed signs.
Insurance: Mathematics and Economics, 2012, accepted.

K. Van Weert, J. Dhaene, and M. Goovaerts.
Optimal portfolio selection for general provisioning and terminal wealth problems.
Insurance: Mathematics and Economics 47(1):90–97, 2010.

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Motivation

- distribution of sum of r.v.
- series of future payments
- stochastic returns
- provisioning or savings context
  - maximize target capital for given probability level
  - maximize probability level s.t. terminal wealth $>$ target wealth

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Motivation

- all payments same sign
  Dhaene, Vanduffel, Goovaerts, Kaas, Vyncke (2005)
- saving-consumption (+ followed by −)
  Vanduffel, Dhaene, Goovaerts (2005)
- more general cash flow patterns?
  e.g., periodic savings and periodic liabilities in pension fund

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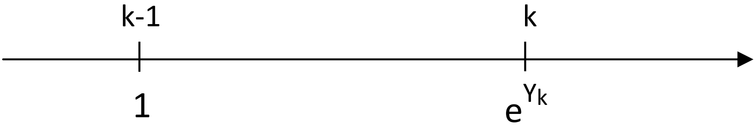
Problem description

- deterministic amounts α_l
- time period and investment horizon: long
- Gaussian model for stochastic returns
-



- i.i.d. normally distributed with mean $\mu - \frac{1}{2}\sigma^2$ and variance σ^2

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Problem description

$$S = \sum_{i=0}^{n} \alpha_i e^{Z_i}, \text{ with } Z_i = \sum_{j=1}^{n} \lambda_{ij} Y_j$$

- $(Y_1, Y_2, \ldots, Y_n)$ multivariate normal
- Problem: impossible to determine distribution of S analytically in closed form
- Convex order bounds: Kaas et al. (2000)
  - $\alpha_i \geq 0$
  - $\mathbb{E}[S \mid \Lambda] = S^\ell \leq_{cx} S \leq_{cx} S^c$, with

$$S^\ell = \sum_{i=0}^{n} \alpha_i e^{\mathbb{E}[Z_i] + \frac{1}{2}(1-r_i^2)\sigma_{Z_i}^2 + r_i \sigma_{Z_i} \Phi^{-1}(U)}$$

$$S^c = \sum_{i=0}^{n} \alpha_i e^{\mathbb{E}[Z_i] + \sigma_{Z_i} \Phi^{-1}(U)}$$

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Problem description

$$\mathbb{E}[S \mid \Lambda] = S^\ell \leq_{cx} S \leq_{cx} S^c$$

- $\alpha_i \geq 0, \forall i \Rightarrow S^c$ and S^ℓ are comonotonic sum
  $\Rightarrow$ straightforward to compute $Q_p[S^c]$ and $Q_p[S^\ell]$

- α_i's with changing signs:
  - $S^c = \sum_{i=0}^{n} \alpha_i e^{\mathbb{E}[Z_i]+\text{sign}(\alpha_i)\sigma_{Z_i}\Phi^{-1}(U)}$
  - Problem: suitable Λ such that S^ℓ is comonotonic sum?

- Vanduffel, Dhaene, Goovaerts (2005): solution for *Saving-Consumption* schemes

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Saving & Terminal Wealth

- Deterministic cash flows $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ with $n \geq 1$:
  - $\alpha_0 > 0$
  - α_i, $0 < i < n$ can be postive or negative
- Generalization of *Saving-Consumption* scheme
- Available surplus at time k:

$$V_k = \sum_{l=0}^{k} \alpha_l e^{Z_{l,k}} = \sum_{l=0}^{k} \alpha_l e^{\sum_{j=l+1}^{k} Y_j}$$

- Terminal wealth:

$$W_n = \max[V_n, 0]$$

- Goal: determine distribution of W_n

Saving & Terminal Wealth

- Approximation:

$$V_n \geq_{cx} V_n^\ell = \mathbb{E}[V_n \mid \Lambda]$$

for some appropriate r.v. Λ

- Maximizing $\mathrm{Var}[V_n^\ell]$ leads to $\Lambda = \sum_{j=1}^n \beta_j Y_j$ with

$$\beta_j = \sum_{l=0}^{j-1} \alpha_l e^{\mathbb{E}[Z_{l,n}] + \frac{1}{2}\sigma^2_{Z_{l,n}}} = \sum_{l=0}^{j-1} \alpha_l e^{(n-l)\mu}$$

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- Distribution of V_n^ℓ:

$$V_n^\ell \stackrel{d}{=} \sum_{l=0}^{n-1} \alpha_l e^{(n-l)\mu - \frac{1}{2} r_l^2 (n-l)\sigma^2 + r_l \sigma \sqrt{n-l}\Phi^{-1}(U)} \equiv f(U)$$

$$\text{with} \quad r_l = \frac{\mathrm{Cov}\left(Z_{l,n}, \Lambda\right)}{\sigma_{Z_{l,n}} \sigma_\Lambda} = \frac{\sum_{j=l+1}^{n} \beta_j}{\sqrt{n-l}\sqrt{\sum_{j=1}^{n} \beta_j^2}}$$

- Distribution of W_n^ℓ:

$$W_n^\ell = \max[V_n^\ell, 0] \stackrel{d}{=} \max[f(U), 0]$$

- If all terms are non-decreasing functions of U, V_n^ℓ is comonotonic sum

$$\Rightarrow Q_p[V_n^\ell] = f(p) = \sum_{l=0}^{n-1} \alpha_l e^{(n-l)(\mu - \frac{1}{2} r_l^2 \sigma^2) + r_l \sigma \sqrt{n-l}\Phi^{-1}(p)}$$

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- If not all terms are non-decreasing functions of U, V_n^ℓ is not comonotonic sum.
- If total sum f is non-decreasing function in interval $(p^\star, 1)$, $Q_p[V_n^\ell]$ can still be used for $p \in (p^\star, 1)$
- conditions on amounts α_l?

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Theorem

If the conditioning random variable Λ equals $\sum_{j=1}^{n} \beta_j Y_j$ with $\beta_j = \sum_{l=0}^{j-1} \alpha_l e^{(n-l)\mu}$, and if the surplus V_l satisfies

$$\mathbb{E}[V_l] > 0, \quad l = 0, \ldots, n-1,$$

then the quantiles of W_n^ℓ are given by

$$Q_p[W_n^\ell] = \max[f(p), 0] = f(p), \quad p^\star < p < 1.$$

The distribution function of W_n^ℓ follows from

$$f(F_{W_n^\ell}(x)) = x, \quad x \geq Q_{p^\star}[W_n^\ell].$$

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Equivalent conditions of $\mathbb{E}[V_l] > 0, \; l = 0, \ldots, n-1$

- $\mathbb{E}[V_l] = e^{\mu}\mathbb{E}[V_{l-1}] + \alpha_l$, $l = 1, \ldots, n-1$, with $\mathbb{E}[V_0] = \alpha_0$:

$$\mathbb{E}[V_l] > 0, \quad \text{for all } l \text{ s.t. } \alpha_l < 0$$

- $\mathbb{E}[V_l] = e^{-(n-l)\mu}\beta_{l+1}$:

$$\beta_{l+1} > 0, \quad l = 0, \ldots, n-1$$

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Sufficient conditions for $\mathbb{E}[V_l] > 0, \; l = 0, \ldots, n-1$

- $\mathbb{E}[V_l] = \sum_{k=0}^{l} \alpha_k e^{(l-k)\mu} = \sum_{k=0}^{l} \alpha_k x^{l-k}$:

$$\mu > \mu^\star \;\Rightarrow\; \mathbb{E}[V_l] > 0, \quad l = 0, \ldots, n-1$$

$$\mu^\star = \max_{l=0,\ldots,n-1} \left(\max \left\{ \mu \mid \mu > 0 \text{ and } \sum_{k=0}^{l} \alpha_k e^{(l-k)\mu} = 0 \right\} \right)$$

- $\mathbb{E}[V_l] = e^{-(n-l)\mu} \beta_{l+1} = e^{-(n-l)\mu} \sum_{k=0}^{l} \alpha_k e^{(n-k)\mu}$

$$\sum_{k=0}^{l} \alpha_k \geq 0 \quad l = 0, \ldots, n-1$$

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Saving & Terminal Wealth

- Example 1 : $n = 20$

10	10	10	10	-40	10	10	10	10	-40
10	10	10	10	-40	10	10	10	10	-40

- Example 2 : $n = 20$

10	10	10	10	-50	10	10	10	10	-50
10	10	10	10	-50	10	10	10	10	-50

$$\mathbb{E}[V_l] > 0, \quad l = 4, 9, 14, 19 \text{ for } \mu \geq 0.088$$

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Saving & Terminal Wealth

- Example 1 : $n = 20$

$$\begin{array}{cccccccccc} 10 & 10 & 10 & 10 & -40 & 10 & 10 & 10 & 10 & -40 \\ 10 & 10 & 10 & 10 & -40 & 10 & 10 & 10 & 10 & -40 \end{array}$$

- Example 2 : $n = 20$

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Saving & Terminal Wealth

How to determine $p^\star$ in

$$Q_p[W_n^\ell] = \max[f(p), 0] = f(p), \quad p^\star < p < 1?$$

Answer: s.t. function f is positive and non-decreasing in p

$$f(p) = \sum_{l=0}^{n-1} \alpha_l e^{(n-l)(\mu - \frac{1}{2} r_l^2 \sigma^2) + r_l \sigma \sqrt{n-l} \Phi^{-1}(p)}$$

1. split $f(p)$ in saving-consumption cases $f_i(p)$ and apply Vanduffel et al. (2005) then

$$p^\star = max_{i \in \{1,\dots,m\}} p_i$$

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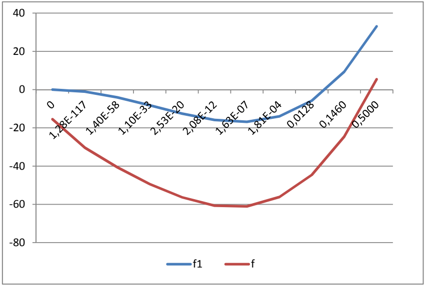
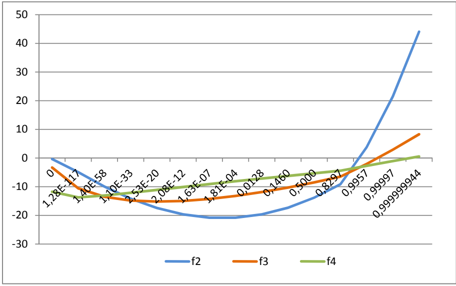
Saving & Terminal Wealth





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$$p^\star = max_{i=1,\dots,4} p_i = p_4 = 0.999999$$

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Saving & Terminal Wealth

Proof that there exists $p^\star \leq \max_{i\in\{1,\dots,m\}} p_i$:

2 use Descartes' rule of sign for generalized polynomials

$$f(p) = h(x) = \sum_{l=0}^{n-1} \alpha_l e^{(n-l)(\mu - \frac{1}{2}\sigma^2)} x^{r_l\sqrt{n-l}}, \quad x = e^{\sigma\Phi^{-1}(p)}$$

$$f'(p) = h'(x) x \frac{\sigma}{\varphi(p)}, \ h'(x) = \sum_{l=0}^{n-1} a_l r_l \sqrt{n-l}\, x^{r_l\sqrt{n-l}-1}$$

$$p^\star = \Phi\left(\frac{1}{\sigma}\ln x^\star\right) \text{ with } x^\star = \max(x_{\max}, x'_{\max})$$

Example 2:

$$(\mu,\sigma) = (0.09, 0.10) \Rightarrow p^\star = 0.437329957844209$$
$$(\mu,\sigma) = (0.10, 0.10) \Rightarrow p^\star = 0.326242535523647$$

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Saving & Terminal Wealth

Proof that there exists $p^\star \leq \max_{i\in\{1,\ldots,m\}} p_i$:

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Reserves for Future Obligations

- Deterministic obligations $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $n \geq 1$:
  - $\alpha_n > 0$
  - α_i, $0 < i < n$ can be postive or negative
- Future obligations at time l:

$$R_l = \sum_{k=l+1}^{n} \alpha_k e^{Z_{l,k}} = \sum_{k=l+1}^{n} \alpha_k e^{-\sum_{j=l+1}^{k} Y_j}$$

- Initial provision:

$$S_0 = \max[R_0, 0]$$

- Goal: determine distribution of S_0

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Reserves for Future Obligations

- Approximation:

$$R_0 \geq_{cx} R_0^\ell = \mathbb{E}[R_0 \mid \Lambda]$$

for some appropriate r.v. Λ

- Maximizing $\mathrm{Var}[R_0^\ell]$ leads to $\Lambda = \sum_{j=1}^n \beta_j Y_j$ with

$$\beta_j = -\sum_{l=j}^n \alpha_l e^{l(-\mu+\sigma^2)}$$

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Reserves for Future Obligations

- Distribution of R_0^ℓ:

$$R_0^\ell \stackrel{d}{=} \sum_{j=1}^{n} \alpha_j e^{-j\mu+(1-\frac{1}{2}r_j^2)j\sigma^2+r_j\sigma\sqrt{j}\Phi^{-1}(U)} \equiv f(U)$$

$$\text{with} \quad r_j = \frac{-\sum_{k=1}^{j}\beta_k}{\sqrt{j}\sqrt{\sum_{k=1}^{j}\beta_k^2}}$$

- Distribution of S_0^ℓ:

$$S_0^\ell = \max[R_0^\ell, 0] \stackrel{d}{=} \max[f(U), 0]$$

- If all terms are non-decreasing functions of U, R_0^ℓ is comonotonic sum

$$\Rightarrow Q_p[R_0^\ell] = f(p) = \sum_{j=1}^{n} \alpha_j e^{-j\mu+(1-\frac{1}{2}r_j^2)j\sigma^2+r_j\sigma\sqrt{j}\Phi^{-1}(p)}$$

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Reserves for Future Obligations

Theorem

If the conditioning random variable Λ equals $\sum_{j=1}^{n} \beta_j Y_j$ with $\beta_j = -\sum_{k=j}^{n} \alpha_k e^{k(-\mu+\sigma^2)}$, and if the obligations R_l satisfies

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The distribution function of S_0^ℓ follows from

$$f(F_{S_0^\ell}(x)) = x, \quad x \geq Q_{p^\star}[S_0^\ell].$$

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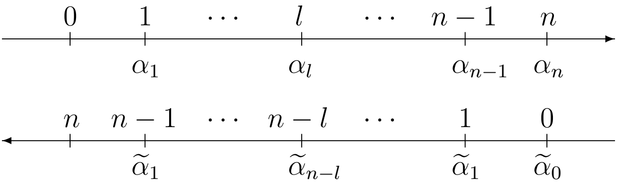
Reserves for Future Obligations

- invert time axis $\Rightarrow$ terminal wealth setting



- $\widetilde{Y}_l = -Y_{n-l+1} \sim \mathcal{N}(\tilde{\mu} - \frac{1}{2}\sigma^2, \sigma^2)$ with $\tilde{\mu} = -\mu + \sigma^2$
  $\tilde{\mu}$ in general negative
- $\widetilde{V}_n = R_0$
- $\mathbb{E}[\widetilde{V}_l] = \alpha_{n-l} + \mathbb{E}[R_{n-l}]$
- $\tilde{f}(U) \stackrel{d}{=} \widetilde{V}_n^{\ell} = R_0^{\ell} \stackrel{d}{=} f(U)$
- $\mathbb{E}[R_l] > 0,\ l = 0, \ldots, n-1 \Rightarrow \sum_{k=j}^{n} \alpha_k \geq 0\ (\mu - \sigma^2 > 0)$

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Thank you for your attention