Continuous time valuation by instantaneous distortion

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Based on joint work with D Madan and M Stadje
DGVFM, 11th Scientific Day
27th April 2012, Stuttgart
Agenda

- Risk measures and non-linear expectations
- Example: distorted expectation
- Continuous valuation and risk measurement
- Examples
Literature on risk measures and non-linear expectations

The literature on risk measures and non-linear expectations is extensive. Key contributions include

▶ Artzner, Delbaen, Eber & Heath (1999),
▶ Carlier & Dana (2004)
▶ Cheridito, Delbaen, Kupper (2004/5/6)
▶ Delbaen (2000/2)
▶ Dennenberg (1994)
▶ Föllmer & Schied (2004)
▶ Fritelli and Rosazza Gianin (2002/3)
▶ Jaschke and Küchler (2001)
▶ Jouini, Schachermayer, Touzi (2005)
▶ Kupper & Schachermayer (2009)
▶ Kusuoka (2001)
▶ Klöppel and Schweizer (2007)
▶ Leitner (2008)
▶ Riedel (2004)
▶ Stadje (2010)
▶ Weber (2006)
▶ Wang

and many more papers (see references therein).
Quantification of risk and risk measurement

Let $X$ be the P&L (Profit and Loss) of a portfolio or position.

**Capital requirement:** the capital $c$ that needs to be added to the risk position such that the risk of

$$X + c$$

is acceptable.

**Risk measure:** a map $\rho$ that adds to $X$ its (minimal) capital requirement $c$:

$$X \mapsto c = \rho(X).$$

**Example:** Value at Risk $\text{VAR}_\alpha$

$$\text{VAR}_\alpha(X) = \sup\{x \in \mathbb{R} : P(X < -x) \geq \alpha\}$$
Quantification of risk and risk measurement

- Let $(\Omega, \mathcal{F}, P)$ be a given probability space.
- **Definition** (CHMP, 2002). A non-linear expectation is a map $\mathcal{E} : L^2(\Omega, \mathcal{F}, P) \to \mathbb{R}$ that satisfies
  - **(Strict monotonicity)**
    
    If $X \leq Y \text{ P-a.s.}$, then $\mathcal{E}(X) \leq \mathcal{E}(Y)$,
    
    and if in addition $\mathcal{E}(X) = \mathcal{E}(Y)$ then $X = Y \text{ P-a.s.}$
  - **(Preservation of constants)**
    
    $\mathcal{E}(c) = c$ for $c \in \mathbb{R}$. 

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  and if in addition \(\mathcal{E}(X) = \mathcal{E}(Y)\) then \(X = Y\) \(P\)-a.s.

- **(Preservation of constants)**

  \[\mathcal{E}(c) = c\] for \(c \in \mathbb{R}\).

**Further properties** A non-linear expectation is ....

- **translation invariant** if \(\mathcal{E}(X + c) = \mathcal{E}(X) + c\) \((c \in \mathbb{R})\).
- **convex** if

  \[\mathcal{E}(\lambda X + (1 - \lambda)Y) \leq \lambda \mathcal{E}(X) + (1 - \lambda)\mathcal{E}(Y)\] \((\lambda \in [0, 1])\).

- **positively homogeneous** if \(\mathcal{E}(\lambda X) = \lambda \mathcal{E}(X)\) \((\lambda \in \mathbb{R}_+)\).
Quantification of risk and risk measurement

Link with risk measures (initiated by ADEH, 1999):

- If \( \mathcal{E} \) is a non-linear expectation that is translation invariant, then
  \[
  \rho(X) = \mathcal{E}(-X)
  \]
  is a monetary risk measure

- If \( \mathcal{E} \) is in addition convex, then \( \rho \) is a convex risk measure

- If \( \mathcal{E} \) is moreover positively homogeneous, then \( \rho \) is a coherent risk measure
Example: distorted expectation

- **Definition:** A distortion $\Xi : [0, 1] \to [0, 1]$ is an increasing function with $\Xi(0) = 0$ and $\Xi(1) = 1$.

- **Definition:** The non-linear expectation $C_{\Xi, \hat{\Xi}}$ associated with the two distortions $\Xi, \hat{\Xi}$ is given by

\[
C_{\Xi, \hat{\Xi}}(X) = \int_0^\infty xd\Xi(F_X(x)) - \int_0^\infty xd\hat{\Xi}(F_{-X}(x))
\]

where $F_X(x) = P(X \leq x)$ and $F_{-X}(x) = P(-X \leq x)$. Here $C_{\Xi, \hat{\Xi}}(X) := -\infty$ if the second integral is infinite.
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A non-linear expectation $\mathcal{E}$ is ....

law-invariant if $\mathcal{E}(X) = \mathcal{E}(Y)$ whenever $X \overset{\mathcal{L}}{=} Y$.
comonotonic if $\mathcal{E}(X + Y) = \mathcal{E}(X) + \mathcal{E}(Y)$ if $X$ and $Y$ are co-monotone.
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- A non-linear expectation $\mathcal{E}$ is ....
  - **law-invariant** if $\mathcal{E}(X) = \mathcal{E}(Y)$ whenever $X \overset{\mathcal{L}}{=} Y$.
  - **comonotonic** if $\mathcal{E}(X + Y) = \mathcal{E}(X) + \mathcal{E}(Y)$ if $X$ and $Y$ are co-monotone.

- If $\Xi$ is concave and $\hat{\Xi}(x) = 1 - \Xi(1 - x)$ then $C^{\Xi, \hat{\Xi}}(X)$ is a convex, comonotonic law-invariant non-linear expectation.
Example: distorted expectation

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- **Definition:** The non-linear expectation $C^{\Xi, \hat{\Xi}}$ associated with the two distortions $\Xi, \hat{\Xi}$ is given by

$$C^{\Xi, \hat{\Xi}}(X) = \int_0^\infty x d\Xi(F_X(x)) - \int_0^\infty x d\hat{\Xi}(F_{-X}(x))$$

where $F_X(x) = P(X \leq x)$ and $F_{-X}(x) = P(-X \leq x)$. Here $C^{\Xi, \hat{\Xi}}(X) := -\infty$ if the second integral is infinite.

- **Kusuoka (2001)** If $\mathcal{E} : L^\infty \rightarrow \mathbb{R}$ is a non-linear expectation that is convex, law-invariant and co-monotonic then

$$\mathcal{E} = C^{\psi, \hat{\psi}}$$

for some concave distortion $\psi$ with $\hat{\psi}(x) := 1 - \psi(1 - x)$. 
Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)\) be a filtered probability space and \(X \in L^2(\Omega, \mathcal{F}, P)\).

A non-linear expectation is called \({\mathcal{F}_t}\)-consistent \((\text{CHMP, 2002})\) if for any \(t \in [0, T]\) there exists a random variable \(Y_t \in L^2(\Omega, \mathcal{F}_t, P)\) such that

\[\mathcal{E}(1_A X) = \mathcal{E}(1_A Y_t) \quad \text{for all } A \in \mathcal{F}_t.\]

Then \(Y_t\) is called the \(\mathcal{F}_t\)-conditional non-linear expectation, and denoted

\[Y_t = \mathcal{E}(X|\mathcal{F}_t).\]
Continuous valuation and risk measurement: Example

Let $U = (U_t)_{t \in [0, T]}$ be given by the Cramér-Lundberg model

$$U_t = x + pt - \sum_{i=1}^{N_t} C_i$$

where

- $C_i$ are IID positive random variables with DF $F$ (claim sizes)
- $x, p > 0$ are constants (initial reserves, premium rate)
- $N$ is a Poisson process

$U$ is defined on a probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)$ denotes the standard filtration generated by $U$.

Let $X \in L^2(\Omega, \mathcal{F}, P)$ be given by $X = H(U_T)$ with $H$ convex.
Continuous valuation and risk measurement: Example

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Continuous valuation and risk measurement: Example

- Let $U$ be given by the Cramér-Lundberg model
  
  $U_t = x + pt - \sum_{i=1}^{N_t} C_i$

- Denote $V(t, x) = \mathbb{E}^{t,x}[X] = \mathbb{E}[H(U_T)|U_t = x]$ and
  
  $\Gamma^X_t(u) = V(t, U_t^- + u) - V(t, U_t^-)$
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$$\Gamma^X_t(u) = V(t, U_{t-} + u) - V(t, U_{t-})$$

- **Definition.** Given two distortions $G, \hat{G}$, the non-linear expectation $\mathcal{E}_{G,\hat{G}}(X)$ is given by

$$\mathcal{E}_{G,\hat{G}}(X) = \mathbb{E}[X] + \mathbb{E} \left[ \lambda \int_0^T C^{G,\hat{G}}(\Gamma^X_s(U)) ds \right]$$

where $C^{G,\hat{G}}(\Gamma^X_s(U))$ is the previously defined non-linear expectation of

$$\Gamma^X_s(U) = \Gamma^X_s(U) - \int_0^\infty \Gamma^X_s(u) F(du) \quad U \sim F.$$
Continuous valuation and risk measurement

Definition. Given two concave distortions $G$, $\hat{G}$, the non-linear expectation $\mathcal{E}_{G,\hat{G}}(X)$ is given by

$$\mathcal{E}_{G,\hat{G}}(X) = \mathbb{E}[X] + \mathbb{E} \left[ \lambda \int_0^T C^{G,\hat{G}}(\Gamma_s(U)) ds \right]$$

where $C^{G,\hat{G}}$ is the previously defined non-linear expectation of

$$\Gamma_s^X(U) = \Gamma_s^X(U) - \int_0^\infty \Gamma_s^X(u) F(du) \quad U \sim F.$$ 

Observe that $V(0, x) = \mathbb{E}[X]$ and

$$V(t, X_t) - \mathbb{E}[X] = \sum_{s \leq t} \Gamma_s^X(\Delta X_s) 1_{\{\Delta X_s \neq 0\}} - \lambda \int_0^t \int_0^\infty \Gamma_s^X(y) F(dy) ds$$

which can be used to define $\Gamma_s^X(u)$. 
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\]

**Definition.** The non-linear expectation \( \mathcal{E} : L^2(\Omega, \mathcal{F}, P) \to \mathbb{R} \) is called **delta law invariant** if \( \mathcal{E}(X) = \mathcal{E}(Y) \) whenever for any \( t \in [0, T] \)

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\Gamma^X_t(U) \overset{\mathcal{L}}{=} \Gamma^Y_t(U).
\]
Definition. Given two concave distortions $G, \hat{G}$, the non-linear expectation $\mathcal{E}_{G,\hat{G}}(X)$ is given by

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Definition. The non-linear expectation $\mathcal{E} : L^2(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is called delta law invariant if $\mathcal{E}(X) = \mathcal{E}(Y)$ whenever for any $t \in [0, T]$

$$\Gamma^X_t(U) \overset{\mathcal{L}}{=} \Gamma^Y_t(U).$$

Theorem. If $\mathcal{E} : L^2(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is a non linear expectation that is convex, comonotonic and delta law invariant then there exist predictable functions $G, \hat{G}$ such that $G_{t,\omega}, \hat{G}_{t,\omega}$ are distortions and

$$\mathcal{E} = \mathcal{E}_{G,\hat{G}}.$$
Continuous valuation and risk measurement: Example

Let $U = (U_t)_{t \in [0, T]}$ be given by the **Samuelson model**

$$dU_t = \mu U_t dt + \sigma U_t dW_t, \quad t \in (0, T], \quad U_0 = x > 0,$$

where

- $W$ is a Brownian motion,
- $\mu \in \mathbb{R}$ is the drift, $\sigma \in \mathbb{R}_+$ the volatility

$U$ is defined on some probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)$ denotes the standard filtration generated by $U$.

Let $X \in L^2(\Omega, \mathcal{F}, P)$ be given by $X = H(U_T)$ with $H$ convex.
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Continuous valuation and risk measurement: Example

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$$
\frac{dU_t}{U_t} = \mu U_t dt + \sigma U_t dW_t, \quad t \in (0, T], \quad U_0 = x > 0,
$$

Denote $V(t, x) = \mathbb{E}^{t, x}[X] = \mathbb{E}[H(U_T) | U_t = x]$ and

$$
\gamma^x_t = \partial_t V_t(U_{t-}) = \frac{\partial V}{\partial x}(t, U_{t-}).
$$
Continuous valuation and risk measurement: Example

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Denote \( V(t, x) = \mathbb{E}^{t, x}[X] = \mathbb{E}[H(U_T)|U_t = x] \) and

\[
\gamma_t^x = \partial V_t(U_{t-}) = \frac{\partial V}{\partial x}(t, U_{t-}).
\]

**Definition.** A drift-shift \( D \) is a non-negative function \( t \mapsto D_t \).
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$$\gamma_t^X = \partial V_t(U_{t-}) = \frac{\partial V}{\partial x}(t, U_{t-}).$$

**Definition.** A drift-shift $D$ is a non-negative function $t \mapsto D_t$.

**Definition.** Given drift-shifts $D, \hat{D}$, the non-linear expectation $E_{D,\hat{D}}(X)$ is given by

$$E_{D,\hat{D}}(X) = E[X] + E\left[\int_0^T \left[\gamma_s^X \right]^+ D_s + \left[-\gamma_s^X \right]^+ \hat{D}_s ds\right]$$

where $x^+ = \max\{x, 0\}$ for any $x \in \mathbb{R}$. 
Continuous valuation and risk measurement: Example

Let $U$ be given by the **perturbed Cramér-Lundberg model**

$$U_t = x + pt + \sigma W_t - \sum_{i=1}^{N_t} C_i$$
Continuous valuation and risk measurement: Example

Let $U$ be given by the perturbed Cramér-Lundberg model

$$U_t = x + pt + \sigma W_t - \sum_{i=1}^{N_t} C_i$$

Definition. Given drift-shifts $D, \hat{D}$ and concave distortions $G, \hat{G}$, the non-linear expectation $\mathcal{E}_{D, \hat{D}, G, \hat{G}}(X)$ of $X$ is given by

$$\mathcal{E}_{D, \hat{D}, G, \hat{G}}(X) = \mathbb{E}[X] + \mathbb{E} \left[ \int_0^T \left[ \gamma_s^X \right]^+ D_s + \left[ -\gamma_s^X \right]^+ \hat{D}_s ds \right]$$

$$+ \mathbb{E} \left[ \lambda \int_0^T C^{G, \hat{G}}(\Gamma_s(U)) ds \right]$$