Continuous time valuation by instantaneous distortion

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Agenda

- Risk measures and non-linear expectations
- Example: distorted expectation
- Continuous valuation and risk measurement
- Examples

Literature on risk measures and non-linear expectations

The literature on risk measures and non-linear expectations is extensive. Key contributions include

- Artzner, Delbaen, Eber & Heath (1999),
- Carlier & Dana (2004)
- Cheridito, Delbaen, Kupper (2004/5/6)
- Coquet, Hu Memin, Peng (2002)
- Delbaen (2000/2)
- Dennenberg (1994)
- Föllmer & Schied (2004)
- Fritelli and Rosazza Gianin (2002/3)
- Jaschke and Küchler (2001)
- Jouini, Schachermayer, Touzi (2005)
- Kupper & Schachermayer (2009)
- Kusuoka (2001)
- Klöppel and Schweizer (2007)
- Leitner (2008)
- Riedel (2004)
- Stadje (2010)
- Weber (2006)
- Wang

and many more papers (see references therein).

Quantification of risk and risk measurement

- Let X be the P&L (Profit and Loss) of a portfolio or position.
- **Capital requirement:** the capital c that needs to be added to the risk position such that the risk of

$$X + c$$

  is acceptable.
- **Risk measure:** a map ρ that adds to X its (minimal) capital requirement c:

$$X \mapsto c = \rho(X).$$

- **Example:** Value at Risk VAR_α

$$VAR_\alpha(X) = \sup\{x \in \mathbb{R} : P(X < -x) \geq \alpha\}$$

Quantification of risk and risk measurement

- Let $(\Omega, \mathcal{F}, P)$ be a given probability space.
- **Definition** (CHMP, 2002). A **non-linear expectation** is a map $\mathcal{E} : L^2(\Omega, \mathcal{F}, P) \to \mathbb{R}$ that satisfies
  - **(Strict monotonicity)**

    $$\text{If } X \leq Y \ P\text{-a.s., then } \mathcal{E}(X) \leq \mathcal{E}(Y),$$

    and if in addition $\mathcal{E}(X) = \mathcal{E}(Y)$ then $X = Y$ P-a.s.
  - **(Preservation of constants)**

    $$\mathcal{E}(c) = c \text{ for } c \in \mathbb{R}.$$

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- **Further properties** A non-linear expectation is ....
  - **translation invariant** if $\mathcal{E}(X + c) = \mathcal{E}(X) + c$ $(c \in \mathbb{R})$.
  - **convex** if

    $$\mathcal{E}(\lambda X + (1-\lambda)Y) \leq \lambda \mathcal{E}(X) + (1-\lambda)\mathcal{E}(Y) \quad (\lambda \in [0,1]).$$

  - **positively homogeneous** if $\mathcal{E}(\lambda X) = \lambda \mathcal{E}(X)$ $(\lambda \in \mathbb{R}_+)$.

Quantification of risk and risk measurement

Link with **risk measures** (initiated by ADEH, 1999):

- If $\mathcal{E}$ is a non-linear expectation that is translation invariant, then
$$\rho(X) = \mathcal{E}(-X)$$
is a **monetary risk measure**
- If $\mathcal{E}$ is in addition convex, then ρ is a **convex** risk measure
- If $\mathcal{E}$ is moreover positively homogeneous, then ρ is a **coherent** risk measure

Example: distorted expectation

- **Definition:** A distortion $\Xi : [0,1] \to [0,1]$ is an increasing function with $\Xi(0) = 0$ and $\Xi(1) = 1$.
- **Definition:** The non-linear expectation $\mathcal{C}^{\Xi,\widehat{\Xi}}$ associated with the two distortions $\Xi, \widehat{\Xi}$ is given by

$$\mathcal{C}^{\Xi,\widehat{\Xi}}(X) = \int_0^{\infty} x \mathrm{d}\Xi(F_X(x)) - \int_0^{\infty} x \mathrm{d}\widehat{\Xi}(F_{-X}(x))$$

where $F_X(x) = P(X \leq x)$ and $F_{-X}(x) = P(-X \leq x)$. Here $\mathcal{C}^{\Xi,\widehat{\Xi}}(X) := -\infty$ if the second integral is infinite.

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- A non-linear expectation $\mathcal{E}$ is ....
  - **law-invariant** if $\mathcal{E}(X) = \mathcal{E}(Y)$ whenever $X \overset{\mathcal{L}}{=} Y$.
  - **comonotonic** if $\mathcal{E}(X+Y) = \mathcal{E}(X) + \mathcal{E}(Y)$ if X and Y are co-monotone.

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- If Ξ is concave and $\widehat{\Xi}(x) = 1 - \Xi(1 - x)$ then $\mathcal{C}^{\Xi,\widehat{\Xi}}(X)$ is a convex, comonotonic law-invariant non-linear expectation.

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- **Kusuoka (2001)** If $\mathcal{E} : L^\infty \to \mathbb{R}$ is a non-linear expecation that is **convex, law-invariant** and **co-monotonic** then

$$\mathcal{E} = \mathcal{C}^{\psi,\widehat{\psi}}$$

for some **concave** distortion ψ with $\widehat{\psi}(x) := 1 - \psi(1-x)$.

Continuous valuation and risk measurement

- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a filtered probability space and $X \in L^2(\Omega, \mathcal{F}, P)$.
- A non-linear expectation is called $\{\mathcal{F}_t\}$-**consistent** (CHMP, 2002) if for any $t \in [0, T]$ there exists a random variable $Y_t \in L^2(\Omega, \mathcal{F}_t, P)$ such that

$$\mathcal{E}(1_A X) = \mathcal{E}(1_A Y_t) \qquad \text{for all } A \in \mathcal{F}_t.$$

- Then Y_t is called the $\mathcal{F}_t$-conditional non-linear expectation, and denoted

$$Y_t = \mathcal{E}(X|\mathcal{F}_t).$$

Continuous valuation and risk measurement: Example

- Let $U = (U_t)_{t \in [0,T]}$ be given by the **Cramér-Lundberg model**

$$U_t = x + pt - \sum_{i=1}^{N_t} C_i$$

  where
  - C_i are IID positive random variables with DF F (claim sizes)
  - $x, p > 0$ are constants (initial reserves, premium rate)
  - N is a Poisson process
- U is defined on a probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)$ denotes the standard filtration generated by U.
- Let $X \in L^2(\Omega, \mathcal{F}, P)$ be given by $X = H(U_T)$ with H convex.

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- Denote $\quad V(t,x) = \mathbb{E}^{t,x}[X] = \mathbb{E}[H(U_T)|U_t = x] \quad$ and

$$\Gamma_t^X(u) = V(t, U_{t-} + u) - V(t, U_{t-})$$

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- **Definition.** Given two distortions $G, \widehat{G}$, the non-linear expectation $\mathcal{E}_{G,\widehat{G}}(X)$ is given by

$$\mathcal{E}_{G,\widehat{G}}(X) = \mathbb{E}[X] + \mathbb{E}\left[\lambda \int_0^T \mathcal{C}^{G,\widehat{G}}(\overline{\Gamma}_s^X(U))\mathrm{d}s\right]$$

where $\mathcal{C}^{G,\widehat{G}}(\overline{\Gamma}_s^X(U))$ is the previously defined non-linear expectation of

$$\overline{\Gamma}_s^X(U) = \Gamma_s^X(U) - \int_0^\infty \Gamma_s^X(u)F(\mathrm{d}u) \qquad U \sim F.$$

Continuous valuation and risk measurement

- **Definition.** Given two concave distortions $G, \widehat{G}$, the non-linear expectation $\mathcal{E}_{G,\widehat{G}}(X)$ is given by

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- Observe that $V(0,x) = \mathbb{E}[X]$ and

$$V(t,X_t) - \mathbb{E}[X] = \sum_{s \le t} \Gamma_s^X(\Delta X_s) 1_{\{\Delta X_s \neq 0\}} - \lambda \int_0^t \int_0^\infty \Gamma_s^X(y)F(\mathrm{d}y)\mathrm{d}s$$

which can be used to define $\Gamma_s^X(u)$.

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- **Definition.** The non-linear expectation $\mathcal{E} : L^2(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is called **delta law invariant** if $\mathcal{E}(X) = \mathcal{E}(Y)$ whenever for any $t \in [0, T]$

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- **Theorem.** If $\mathcal{E} : L^2(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is a non linear expectation that is convex, comonotonic and delta law invariant then there exist predictable functions $G, \widehat{G}$ such that $G_{t,\omega}, \widehat{G}_{t,\omega}$ are distortions and

$$\mathcal{E} = \mathcal{E}_{G,\widehat{G}}.$$

Continuous valuation and risk measurement: Example

- Let $U = (U_t)_{t\in[0,T]}$ be given by the **Samuelson model**

$$\mathrm{d}U_t = \mu U_t \mathrm{d}t + \sigma U_t \mathrm{d}W_t, \qquad t \in (0,T], \quad U_0 = x > 0,$$

  where
  - W is a Brownian motion,
  - $\mu \in \mathbb{R}$ is the drift, $\sigma \in \mathbb{R}_+$ the volatility
- U is defined on some probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)$ denotes the standard filtration generated by U.
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- Denote $\quad V(t, x) = \mathbb{E}^{t,x}[X] = \mathbb{E}[H(U_T)|U_t = x] \quad$ and

$$\Upsilon_t^X = \partial V_t(U_{t-}) = \frac{\partial V}{\partial x}(t, U_{t-}).$$

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$$\mathcal{E}_{D,\widehat{D}}(X) = \mathbb{E}[X] + \mathbb{E}\left[\int_0^T \left[\Upsilon_s^X\right]^+ D_s + \left[-\Upsilon_s^X\right]^+ \widehat{D}_s \mathrm{d}s\right]$$

where $x^+ = \max\{x, 0\}$ for any $x \in \mathbb{R}$.

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$$U_t = x + pt + \sigma W_t - \sum_{i=1}^{N_t} C_i$$

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- **Definition.** Given drift-shifts $D, \widehat{D}$ and concave distortions $G, \widehat{G}$, the non-linear expectation $\mathcal{E}_{D,\widehat{D},G,\widehat{G}}(X)$ of X is given by

$$\begin{aligned} \mathcal{E}_{D,\widehat{D},G,\widehat{G}}(X) &= \mathbb{E}[X] + \mathbb{E}\left[\int_0^T \left[\Upsilon_s^X\right]^+ D_s + \left[-\Upsilon_s^X\right]^+ \widehat{D}_s \mathrm{d}s\right] \\ &+ \mathbb{E}\left[\lambda \int_0^T \mathcal{C}^{G,\widehat{G}}(\overline{\Gamma}_s(U))\mathrm{d}s\right] \end{aligned}$$