The Crash-NIG Copula model: modelling dependence in credit portfolios through the crisis

Anna Schlösser
Risklab

Rudi Zagst
TU München

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Overview

- Goal of our research
- Collateralized Debt Obligations
- One-factor copula model for credit portfolios
- Model extensions
- Conclusion

Goal of our research

- Credit portfolio model for risk management applications:

  → simulation of rating transition and default of credit portfolio
  → re-pricing of CDO tranches along the paths
  → calculation of risk measures / portfolio optimization

- Model requirements:

  → fast (semi-analytical CDO tranche valuation)
  → arbitrage-free (consistent modelling of different tranches and maturities)
  → stylized facts (correlation regimes)

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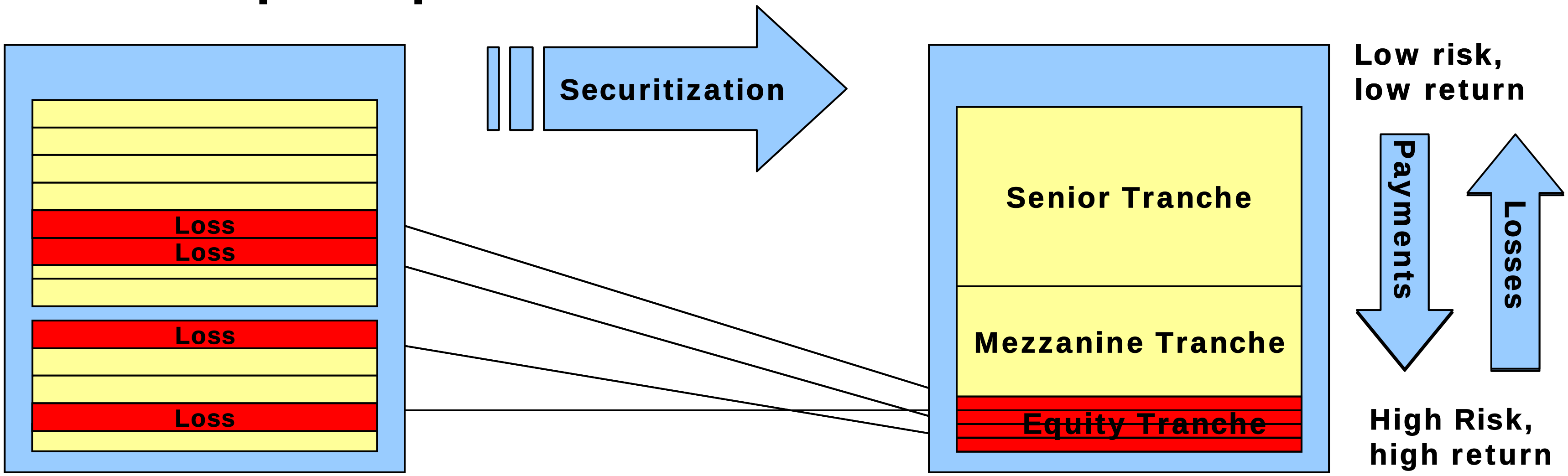
Collateralized Debt Obligations: tranche waterfall

- **Waterfall principle:**



- Investors receive payments according to the seniority of their tranche beginning with the most senior tranche.
- Losses are distributed in reverse order.
- Investors receive a higher coupon for the less senior tranches to compensate higher risk.

CDOs: general valuation framework

Tranche j:

- l_j – lower attachment point
- u_j – upper attachment point
- s_j – annualized spread

- Portfolio loss L(t) → the loss for tranche j

$$L_j(t) = (\min\{ L(t), u_j\} - l_j)^+, \ t \in [0,T].$$

- The expected discounted premium leg

$$PL_j = s_j \cdot \sum_{k=1}^{n} e^{-r \cdot t_k} \cdot \Delta t_k \cdot (u_j - l_j - E[L_j(t_k)])$$

- The expected discounted default leg

$$DL_j = \sum_{k=1}^{n} e^{-r \cdot t_k} \cdot (E[L_j(t_k)] - E[L_j(t_{k-1})])$$

CDOs: general valuation framework

Calculation of the expected tranche loss given the distribution function of the relative portfolio loss F(t,x) and the recovery rate R:

$$E[L_j(t_k)] = (1-R)\cdot\left(\int_{\frac{l_j}{1-R}}^{1}\left(x-\frac{l_j}{1-R}\right)dF(t,x) - \int_{\frac{u_j}{1-R}}^{1}\left(x-\frac{u_j}{1-R}\right)dF(t,x)\right)$$

Problem:
Derivation of the distribution function of the relative portfolio loss F(t,x).

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One-factor copula models for credit portfolios

General distributions

Definition (one-factor copula model):

- Standardized asset return up to time t of the i-th issuer in the portfolio:

$$A_i(t) = a_i \cdot M(t) + \sqrt{1-a_i^2} \cdot X_i(t)$$

  M(t) and $X_i(t)$, i=1,...,m, are independent random variables.

- Known distribution functions: $F_M(t,\cdot)$, $F_X(t,\cdot)$ of $X_i(t)$, and $F_A(t,\cdot)$ of $A_i(t)$.
- The variable $A_i(t)$ is mapped to default time τ_i of the i-th issuer using a percentile-to-percentile transformation, i.e. the issuer i defaults before time t when

$$A_i(t) \le F_A^{-1}(t, Q(t)) = C(t)$$

- $Q_i(t)=Q(t)$ is the (risk-neutral) probability of the issuer i=1,...,m to default before time t.
- Q(t) is estimated from the average CDS spread.

One-factor copula models for credit portfolios

General distributions: portfolio loss

Theorem 1 (One-factor copula model):

Loss distribution of a **large homogeneous portfolio**

$$F_\infty(t,x) = 1 - F_M\left(t, \frac{F_A^{-1}(t,Q(t)) - \sqrt{1-a^2}\cdot F_X^{-1}(t,x)}{a}\right)$$

with $x \in [0,1]$ denoting the relative portfolio loss and Q(t) denoting the risk-neutral default probability of each issuer in the portfolio.

Remark (One-factor Gaussian copula model, Vasicek (1987,1991)):

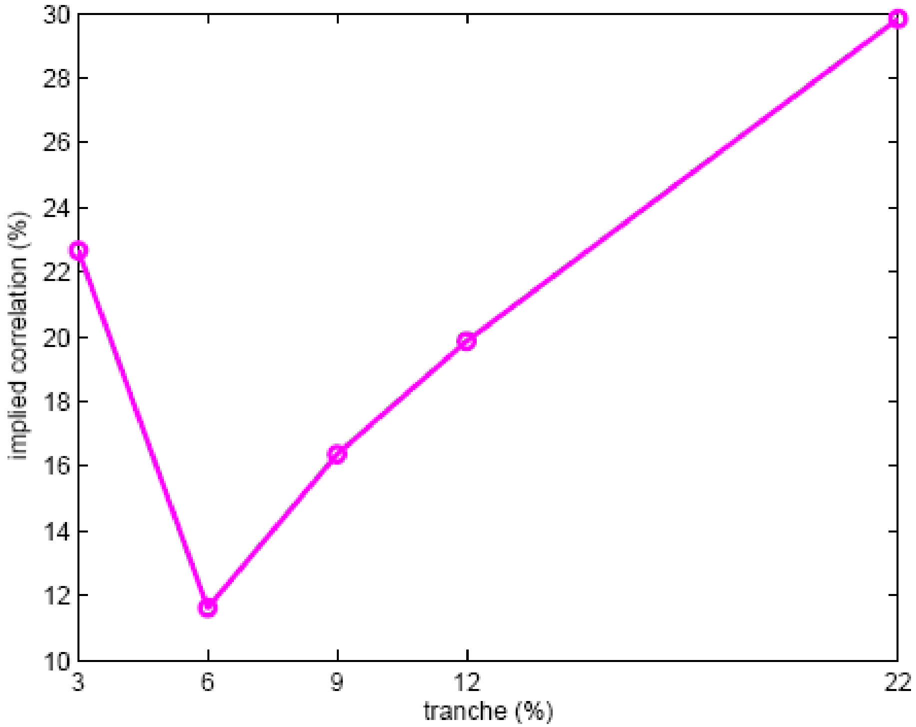
Gaussian distribution for all factors:

$$F_\infty(t,x) = \Phi\left(\frac{\sqrt{1-a^2}\cdot\Phi^{-1}(x) - \Phi^{-1}(Q(t))}{a}\right) = \Phi\left(\frac{\sqrt{1-a^2}\cdot\Phi^{-1}(x) - C(t)}{a}\right)$$

One-factor copula models for credit portfolios

The problems with the one-factor Gaussian copula model

- **Problem:** It is impossible to fit all tranches with the same implied correlation (correlation smile)



- Different implied correlations for tranches with different maturities.

One-factor copula model for credit portfolios

Literature on solving the problem

Alternative distributions and copulas:

- O'Kane & Schloegel (2003): Student-t copula
- Schönbucher (2003): Archimedian copulas
- Andersen & Sidenius (2005): Marshall-Olkin copula
- Hull & White (2004): Double-t copula

Additional stochastic factors:

- Andersen & Sidenius (2005): Random recovery & random correlation factor
- Hull et al (2005): Random correlation correlated with the market factor
- Trinh et al (2005): Idiosyncratic and systematic jumps to default

→ **Empirical comparison:** Burtschell et al (2009): Double-t copula

→ **Heavy-tailed distribution with better properties:**
Kalemanova et al (2007): NIG factor copula

One-factor copula model for credit portfolios

NIG copula model (Kalemanova, Schmid, Werner (2007) Journal of derivatives)

- $NIG_{(s)} = NIG\left(s\cdot\alpha, s\cdot\beta, -s\cdot\frac{\beta\cdot\gamma^2}{\alpha^2}, s\cdot\frac{\gamma^3}{\alpha^2}\right)$ with corresponding distribution function $F_{NIG_{(s)}}(x)$ and $\gamma = \sqrt{\alpha^2 - \beta^2}$.

- $M(t) \sim NIG_{(1)}$, $X_i(t) \sim NIG_{\left(\frac{\sqrt{1-a^2}}{a}\right)}$

- Then, $A_i(t) \sim NIG_{\left(\frac{1}{a}\right)}$ (due to stability under convolution of NIG distribution)

- and the distribution function of the portfolio loss at time t is given by

$$F_\infty(t,x) = 1 - F_{NIG_{(1)}}\left(\frac{F^{-1}_{NIG_{\left(\frac{1}{a}\right)}}(Q(t)) - \sqrt{1-a^2}\cdot F^{-1}_{NIG_{\left(\frac{\sqrt{1-a^2}}{a}\right)}}(x)}{a}\right).$$

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One-factor copula model for credit portfolios
Empirical comparison

- Calibration of the iTraxx tranches on 12.04.2006
-

	Market	Gaussian	t(4)-t(4)	t(3)- t(3)	NIG(1)	NIG(2)
0-3%	23,53%	23,53%	23,53%	23,53%	23,53%	23,53%
3-6%	62,75 bp	140,46 bp	73,3 bp	53,88 bp	62,75 bp	62,75 bp
6-9%	18 bp	29,91 bp	28,01 bp	23,94 bp	27,9 bp	27,76 bp
9-12%	9,25 bp	7,41 bp	16,53 bp	15,96 bp	17,64 bp	17,42 bp
12-22%	3,75 bp	0,8 bp	8,68 bp	9,94 bp	9,79 bp	9,6 bp
absolute error		94,41 bp	32,82 bp	27,82 bp	24,34 bp	23,77 bp
correlation		15,72%	19,83%	18,81%	16,21%	15,94%
α					0,4794	0,6020
β					0	-0,1605
comp. time		0,5 s	12,6 s	11 s	1,5 s	1,6 s

- t(n)-t(n) is a double-t distribution with n degrees of freedom
- NIG(1) is the NIG model with β=0 and thus only one parameter α
- NIG(2) is the NIG model with two parameters, α and β.

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Model extensions - Overview

Schlösser, Zagst "The Crash-NIG Copula model: modelling dependence in credit portfolios through the crisis"

1. **Term structure extension**

   → Consistent modeling of portfolio loss distributions over different time horizons

2. **Large Homogeneous Cells (**Desclee et al (2006) for the Gaussian copula model**)**

   → Relaxing the assumption of large homogeneous portfolio and modeling different ratings and rating transitions

3. **Regime-switching correlation**

   → Allow several correlation regimes with HMM, but still keep the model semi-analytically tractable and fast

Model extensions

Term structure extension: NIG process

- The appropriate process for the factors with NIG-distributed increments is given by $N_{(s)}(t)$ with a scaling factor s, independent increments

$$dN_{(s)}(t) \sim NIG\left(s\cdot\alpha, s\cdot\beta, -s\cdot\frac{\beta\cdot\gamma^2}{\alpha^2}dt, s\cdot\frac{\gamma^3}{\alpha^2}dt \right) =: \mathrm{NIG}_{(s)}(\mathrm{dt}),\ \gamma=\sqrt{\alpha^2-\beta^2},$$

and the following properties:

1. The increments have zero mean and variance dt.
2. The process has zero mean, variance t, skewness $3\cdot\frac{\beta}{s\cdot\gamma^2\cdot\sqrt{t}}$ and kurtosis $3+3\cdot\left(1+4\cdot\left(\frac{\beta}{\alpha}\right)^2\right)\cdot\frac{\alpha^2}{s^2\cdot\gamma^4\cdot t}$.
3. $N_{(s)}(t) \sim NIG\left(s\cdot\alpha, s\cdot\beta, -s\cdot\frac{\beta\cdot\gamma^2}{\alpha^2}\cdot t, s\cdot\frac{\gamma^3}{\alpha^2}\cdot t \right) =: \mathrm{NIG}_{(s)}(\mathrm{t})$.

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Model extensions
Large homogeneous cells

- Calibration to iTraxx data for the 12th of April 2006:

	Maturity	0-3%	3-6%	6-9%	9-12%	12-22%	error		parameter
Market	5	23,53%	62,75 bp	18 bp	9,25 bp	3,75 bp			
	7	36,875%	189 bp	57 bp	26,25 bp	7,88 bp			
	10	48,75%	475 bp	124 bp	56,5 bp	19,5 bp			
Gaussian LHC	5	28,85%	92,02 bp	32,70 bp	13,74 bp	2,76 bp	49,44 bp	a_{AAA}	0,6052
	7	53,43%	198,81 bp	71,91 bp	32,88 bp	7,88 bp	30,85 bp	a_{AA}	0,0004
	10	63,19%	445,90 bp	133,39 bp	65,30 bp	18,42 bp	48,37 bp	a_A	0,7211
							128,66 bp	a_{BBB}	0,0005
term-structure NIG(1) LHC	5	24,92%	58,42 bp	23,4 bp	14,25 bp	7,59 bp	18,61 bp	a_{AAA}	0,4217
	7	48,19%	202,08 bp	53,31 bp	27,08 bp	12,05 bp	22,27 bp	a_{AA}	0,5139
	10	56,09%	475,00 bp	124,00 bp	51,93 bp	18,87 bp	5,20 bp	a_A	0,4522
							46,09 bp	a_{BBB}	0,2598
								α	0,2269

	AAA	AA	A	BBB
portfolio weight	0,8 %	10,4%	42,4%	46,4 %

Model extensions

Regime-switching extension: motivation and model requirements

- **One correlation regime** is enough for pricing purposes ↔ unrealistic for simulating future scenarios.
- Correlation as a **stochastic process** does not fit to the concept of the factor copula models.
- **Several correlation regimes** are sufficient for a simulation-based risk measurement framework.

- The **Crash-NIG model** having two different correlation states has to satisfy the following requirements:
  1. The distributions of both factors in different states are stable under convolution.
  2. The asset return has the same distribution in both states to ensure an easy derivation of the default thresholds.
  3. The distributions of the factors in both states have zero mean.
  4. The distribution of the market factor does not depend on the correlation.

Model extensions

Crash-NIG copula model

Theorem 2 (Crash-NIG copula model): The Crash-NIG model is given by the asset return up to time t of the i-th issuer in cell j=1,...,J, $A_{ij}(t)$, of the form:

$$dA_{ij}(t) = a_j \cdot M(t) + \sqrt{1-a_j^2} \cdot X_{ij}(t)$$

with independent stochastic processes:

$$dM(t) \sim NIG_{(1)}\left(\Lambda_t^2\, dt\right),\ dX_{ij}(t) \sim NIG_{\left(\frac{\sqrt{1-a_j^2}}{a_j}\right)}\left(\frac{1-\Lambda_t^2 \cdot a_j^2}{1-a_j^2}\, dt\right),$$

where Λ_t is a Markov process with state space $\{1,\lambda\}$, initial distribution π, and transition function $\{P(h)\}_{h\geq 0}$. The distribution of the increment of the asset return is

$$dA_{ij}(t) \sim NIG_{\left(\frac{1}{a_j}\right)}(dt).$$

Model extensions
Crash-NIG copula model

Remarks:

- In the first correlation regime, the variance of all factor changes is dt.
- The variance of the factors in the second regime is

$$Var(dM) = \lambda^2 \, dt \, , \; Var(dX_{ij}) = \frac{1 - \lambda^2 \cdot a_j^2}{1 - a_j^2} \, dt \, .$$

- Thus, the correlation of asset returns of an issuer i_1 from the rating cell j_1 and an issuer i_2 from the rating cell j_2 in the second regime is

$$Corr\left[dA_{i_1 j_1}(t), dA_{i_2 j_2}(t) \right] = \frac{a_{j_1} \cdot a_{j_2} \cdot Var\left[dM(t) \right]}{dt} = a_{j_1} \cdot a_{j_2} \cdot \lambda^2 \, .$$

- The higher λ, the higher the correlation and the higher the systemic risk.
- The model is straightforward for more than two regimes.

Model extensions
Crash-NIG copula model: valuation

- There is no analytical expression for the unconditional distributions of the factors, but they can be approximated by moment matching with NIG distribution.
  - simple approximation with two moment matching
  - more exact approximation with four moment matching

- Semi-analytical formulas for portfolio loss distribution available

Model extensions
Crash-NIG copula model: calibration

Data:

- The history of the iTraxx Europe tranched index since its origination on the 21th of June 2004 until the 6th of May 2008 (from “Morgan Markets”).

Set-up:

- **Three-state model**: the two crisis (market turbulences after the downgrade of Ford and General Motors in Mai 2005 and the credit crisis starting in July 2007) have different character.

- The iTraxx tranches market is more liquid than the single-CDS market → **liquidity indicators l_r** , with r=1,2,3 the current state:
  - $(1-l_r)\cdot s$ liquidity spread part
  - $l_r \cdot s$ credit spread part.

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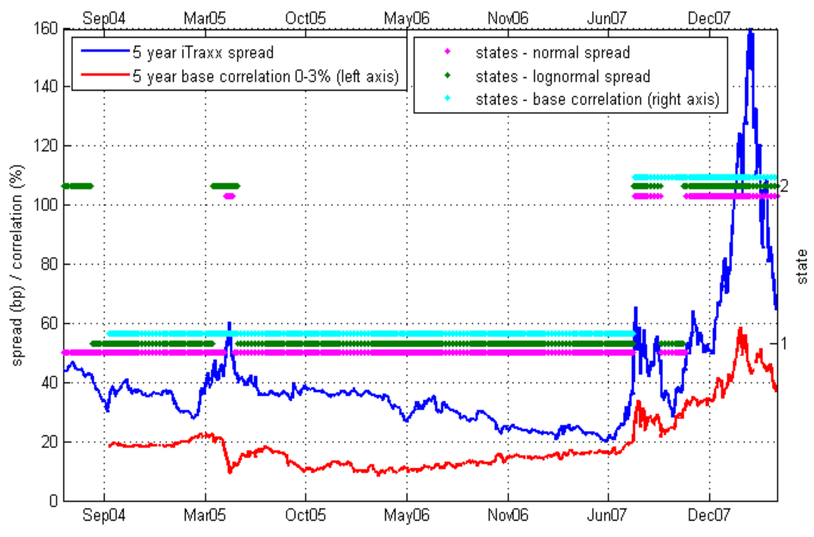
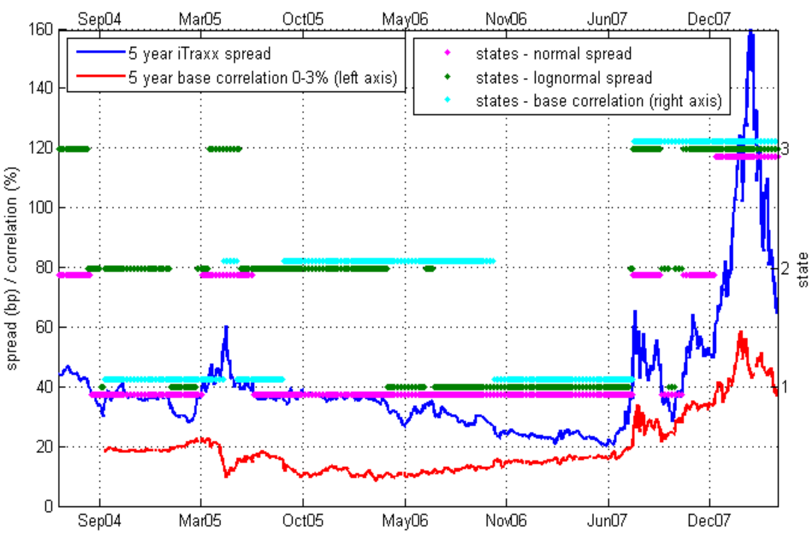
Model extensions
Crash-NIG copula model: calibration

Two-step calibration:

1. Hidden Markov Model (HMM) is estimated separately using the 5-year iTraxx index spreads under assumption of normal distribution:
   – Transition matrix estimated with Baum-Welch algorithm
   – Most likely states estimated with Viterbi algorithm.



Calibration of the HMM with two states



Calibration of the HMM with three states

Model extensions
Crash-NIG copula model: calibration

Two-step calibration:

2. Other NIG model parameters (a_j, λ, α and l_r) are calibrated to the tranches' history:

Parameter	Crash-NIG model (3 states, liquidity)	Crash-NIG model (2 states, liquidity)	NIG model (1 state, liquidity)	Crash-NIG model (2 states, no liquidity)
alpha	0,3274	0,3957	0,3615	0.1717
a1	0,2562	0,1680	0,2476	0,3869
a2	0,5437	0,4275	0,9607	0,4493
a3	0,3429	0,4275	0,4975	0,4494
a4	0,2130	0,1767	0,3256	0,2820
a5	0,0828	0,1705	0,1161	0,2541
lambda_1	0,2353	2,2220		2,1141
lambda_2	1,7443			
l_1	0,9679	0,9439	0,9562	
l_2	0,8827	0,7330		
l_3	0,7361			
Aver. Error (%)	14,8	18,13	23,98	25,76

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Conclusion

- The Crash-NIG copula model allows to consistently model:
  - → CDO tranches with different attachment/detachment points
  - → CDO tranches with different maturities
  - → rating composition of the portfolio
  - → different correlation regimes
  - → liquidity premium.

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Conclusion

Anna Schlösser, Rudi Zagst „The Crash-NIG copula model: risk measurement and management of credit portfolios." Journal of Risk Management in Financial Institutions, Volume 4, Number 4, 2011.

→ joint Monte Carlo simulation of rating transitions with the Crash-NIG copula model with other risk factors (credit spreads, interest rates)

→ re-pricing of CDO tranches along the paths with the Crash-NIG copula model

→ scenario-based optimization of balanced portfolios including structured credit instruments.

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Thank you for your attention!