Local Volatility Pricing Models for Long-Dated Derivatives in Finance and Insurance

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Outline of the talk

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3 The model for GAO’s
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Recent years, the long-dated (maturity > 1 year) foreign exchange (FX) option’s market has grown considerably

- Vanilla options (European Call and Put)
- Exotic options (barriers, ...)
- Hybrid options (PRDC swaps)

See e.g.

A suitable pricing model for long-dated FX options has to take into account the risks linked to:

- domestic and foreign interest rates
  - by using stochastic processes for both domestic and foreign interest rates
    \[
    dr_d(t) = [\theta_d(t) - \alpha_d(t)r_d(t)]dt + \sigma_d(t)dW_d^{DRN}(t),
    \]
    \[
    dr_f(t) = [\theta_f(t) - \alpha_f(t)r_f(t) - \rho_{sf}\sigma_f(t)\sigma(t, S(t))]dt + \sigma_f(t)dW_f^{DRN}(t)
    \]
- the volatility of the spot FX rate (Smile/Skew effect)
  - by using a local volatility \(\sigma(t, S(t))\) for the FX spot
    \[
    dS(t) = (r_d(t) - r_f(t))S(t)dt + \sigma(t, S(t))S(t)dW_S^{DRN}(t),
    \]
  - by using a stochastic volatility \(\nu(t)\) for the FX spot
    \[
    dS(t) = (r_d(t) - r_f(t))S(t)dt + \sqrt{\nu(t)}S(t)dW_S^{DRN}(t),
    \]
    \[
    d\nu(t) = \kappa(\theta - \nu(t))dt + \xi \sqrt{\nu(t)}dW_{\nu}^{DRN}(t)
    \]
- and/or jump
Stochastic volatility models with stochastic interest rates:


Local volatility models with stochastic interest rates:


Local volatility models for equity: Derman & Kani (1994), Dupire (1994), ...

- the local volatility \( \sigma(t, S(t)) \) is a deterministic function of both the spot and time.
  - It avoids the problem of working in incomplete markets in comparison with stochastic volatility models and is therefore more appropriate for hedging strategies.
- has the advantage to be calibrated on the complete implied volatility surface.
  - local volatility models usually capture more precisely the surface of implied volatilities than stochastic volatility models.
- Also some disadvantages, depending on the situation.
The calibration of a model is usually done on the vanilla option market → local and stochastic volatility models (well calibrated) return the same price for these options.

But calibrating a model to the vanilla market is by no means a guarantee that all types of options will be priced correctly

**example:** We have compared short-dated barrier option market prices with the corresponding prices derived from either a Dupire local volatility or a Heston stochastic volatility model both calibrated on the vanilla smile/skew.
A FX market characterized by a mild skew (USDCHF) exhibits mainly a stochastic volatility behavior,

A FX market characterized by a dominantly skewed implied volatility (USDJPY) exhibits a stronger local volatility component.

See Bossens et al. (2010), IJTAF.
The market dynamics could be better approximated by a hybrid volatility model that contains both stochastic volatility dynamics and local volatility ones.

Lipton (2002), Lipton and McGhee (2002), Madan et al. (2007) example:

\[
\begin{align*}
    dS(t) &= (r_d(t) - r_f(t))S(t)dt + \sigma_{LOC2}(t, S(t))\sqrt{\nu(t)}S(t)dW_S^{DRN}(t), \\
    dr_d(t) &= [\theta_d(t) - \alpha_d(t)r_d(t)]dt + \sigma_d(t)dW_d^{DRN}(t), \\
    dr_f(t) &= [\theta_f(t) - \alpha_f(t)r_f(t) - \rho_fS\sigma_f(t)\sigma(t, S(t))]dt + \sigma_f(t)dW_f^{DRN}(t), \\
    d\nu(t) &= \kappa(\theta - \nu(t))dt + \xi\sqrt{\nu(t)}dW_\nu^{DRN}(t).
\end{align*}
\]

The local volatility function \( \sigma_{LOC2}(t, S(t)) \) can be calibrated from the local volatility that we have in a pure local volatility model!
The model for FX Derivatives
The model for FX Derivatives

The three-factor model with local volatility

- The spot FX rate $S$ is governed by the following dynamics

$$dS(t) = (r_d(t) - r_f(t))S(t)dt + \sigma(t, S(t))S(t)dW^\text{DRN}_S(t),$$

\begin{align}
\theta_d(t) - \alpha_d(t)r_d(t) & \quad \text{for } dr_d(t) = [\theta_d(t) - \alpha_d(t)r_d(t)]dt + \sigma_d(t)dW^\text{DRN}_d(t), \\
\theta_f(t) - \rho_f \sigma_f(t) & \quad \text{for } dr_f(t) = [\theta_f(t) - \alpha_f(t)r_f(t) - \rho_f \sigma_f(t)]dt + \sigma_f(t)dW^\text{DRN}_f(t),
\end{align}

- domestic and foreign interest rates $r_d$ and $r_f$ follow a Hull-White one factor Gaussian model defined by the Ornstein-Uhlenbeck processes

$$\left(\begin{array}{ccc}
1 & \rho_{Sd} & \rho_{Sf} \\
\rho_{Sd} & 1 & \rho_{df} \\
\rho_{Sf} & \rho_{df} & 1
\end{array}\right).$$
Forward PDE for the probability density

References:
Dupire (1994), Derman and Kani (1994),...
Overhaus et al. (2006) ‘Equity Hybrid Derivatives’
Consider a derivative that pays off $V(S(t), r_d(t), r_f(t), t)$ at time $t$.

$$V(S(0), r_d(0), r_f(0), t = 0) = \mathbf{E}^{Q_d}[e^{-\int_0^t r_d(s)ds}V(S(t), r_d(t), r_f(t), t)]$$

$$= P_d(0, t)\mathbf{E}^{Q_t}[V(S(t), r_d(t), r_f(t), t)]$$

$$= P_d(0, t) \int \int \int V(x, y, z, t)\phi_F(x, y, z, t)dxdydz \quad (4)$$

where $Q_t$ is the $t$-forward neutral probability and $\phi_F(x, y, z, t)$ corresponds to the $t$-forward probability density.
Following the same approach as in 'Equity Hybrid derivatives' (Overhaus et al., 2006), one derives the following Fokker-Plank equation for \( \phi_F(x, y, z, t) \):

\[
0 = \frac{\partial \phi_F}{\partial t} + (r_d(t) - f_d(0, t)) \phi_F + \frac{\partial[(r_d(t) - r_f(t))S(t)\phi_F]}{\partial x} + \frac{\partial[(\theta_d(t) - \alpha_d(t) r_d(t))\phi_F]}{\partial y} \\
+ \frac{\partial[(\theta_f(t) - \alpha_f(t) r_f(t) - \rho_{fs} \sigma_f(t) \sigma(t, S(t)))\phi_F]}{\partial z} - \frac{1}{2} \frac{\partial^2[\sigma^2(t, S(t))S^2(t)\phi_F]}{\partial x^2} \\
- \frac{1}{2} \frac{\partial^2[\sigma^2_d(t)\phi_F]}{\partial y^2} - \frac{1}{2} \frac{\partial^2[\sigma^2_f(t)\phi_F]}{\partial z^2} - \frac{\partial^2[\sigma(t, S(t))S(t)\sigma_d(t)\rho_{sd}\phi_F]}{\partial x \partial y} \\
- \frac{\partial^2[\sigma(t, S(t))S(t)\sigma_f(t)\rho_{sf}\phi_F]}{\partial x \partial z} - \frac{\partial^2[\sigma_d(t)\sigma_f(t)\rho_{df}\phi_F]}{\partial y \partial z}.
\]  

This PDE is solved forward in time with the initial condition at time \( t = 0 \) given by

\( \phi_F(x, y, z, t) = \delta(x - x_0, y - y_0, z - z_0) \), where \( \delta \) is the Dirac delta function and \( x_0, y_0 \) and \( z_0 \) correspond to the values at time \( t = 0 \) of the spot FX rate, the domestic and foreign interest rates respectively.
The local volatility derivation: standard approach

References:
Dupire (1994), Derman and Kani (1994),...
Overhaus et al. (2006) ‘Equity Hybrid Derivatives’

Other approach: Atlan (2006) by using Tanaka’s formula
The local volatility derivation: standard approach

- Consider the forward call price \( \tilde{C}(K, t) \) of strike \( K \) and maturity \( t \), defined (under the \( t \)-forward measure \( Q_t \)) by

\[
\tilde{C}(K, t) = \frac{C(K, t)}{P_d(0, t)} = E^{Q_t}[(S(t) - K)^+] = \int \int \int_{K}^{+\infty} (S(t) - K) \phi_F(S, r_d, r_f, t) dS dr_d dr_f.
\]

- Differentiating it with respect to the maturity \( t \) leads to

\[
\frac{\partial \tilde{C}(K, t)}{\partial t} = \int \int \int_{K}^{+\infty} (S(t) - K) \frac{\partial \phi_F(S, r_d, r_f, t)}{\partial t} dS dr_d dr_f
\]

- we have shown that the \( t \)-forward probability density \( \phi_F \) satisfies the following forward PDE:

\[
\frac{\partial \phi_F}{\partial t} = -(r_d(t) - f_d(0, t)) \phi_F - \frac{\partial [(r_d(t) - r_f(t)) S(t) \phi_F]}{\partial x} - \frac{\partial [(\theta_d(t) - \alpha_d(t) r_d(t)) \phi_F]}{\partial y}
\]

\[
- \frac{\partial [(\theta_f(t) - \alpha_f(t) r_f(t)) \phi_F]}{\partial z} + \frac{1}{2} \frac{\partial^2 [\sigma^2(t, S(t)) S^2(t) \phi_F]}{\partial x^2} - \frac{1}{2} \frac{\partial^2 [\sigma^2_d(t) \phi_F]}{\partial y^2} + \frac{1}{2} \frac{\partial^2 [\sigma^2_f(t) \phi_F]}{\partial z^2}
\]

\[
+ \frac{\partial^2 [\sigma(t, S(t)) S(t) \sigma_d(t) \rho_S \phi_F]}{\partial x \partial y} + \frac{\partial^2 [\sigma(t, S(t)) S(t) \sigma_f(t) \rho_S \phi_F]}{\partial x \partial z} + \frac{\partial^2 [\sigma_d(t) \sigma_f(t) \rho_d \phi_F]}{\partial y \partial z}.
\]
The local volatility derivation: standard

Integrating by parts several times we get

\[
\frac{\partial \tilde{C}(K, t)}{\partial t} = f_d(0, t)\tilde{C}(K, t) + \int \int \int_{K}^{+\infty} [r_d(t)K - r_f(t)S(t)]\phi_F(S, r_d, r_f, t) dS d\rho d\sigma \\
+ \frac{1}{2} (\sigma(t, K)K)^2 \int \int \phi_F(K, r_d, r_f, t) dr d\sigma \\
= f_d(0, t)\tilde{C}(K, t) + \mathbb{E}^Q_t [(r_d(t)K - r_f(t)S(t))1\{S(t) > K\}] \\
+ \frac{1}{2} (\sigma(t, K)K)^2 \frac{\partial^2 \tilde{C}(K, t)}{\partial K^2}.
\]

This leads to the following expression for the local volatility surface in terms of the forward call prices $\tilde{C}(K, t)$

\[
\sigma^2(t, K) = \frac{\frac{\partial \tilde{C}(K, t)}{\partial t} - f_d(0, t)\tilde{C}(K, t) - \mathbb{E}^Q_t [(r_d(t)K - r_f(t)S(t))1\{S(t) > K\}]}{\frac{1}{2}K^2 \frac{\partial^2 \tilde{C}(K, t)}{\partial K^2}}.
\]
The (partial) derivatives of the forward call price with respect to the maturity can be rewritten as

\[
\frac{\partial \tilde{C}(K, t)}{\partial t} = \frac{\partial [\frac{C(K, t)}{P_d(0, t)}]}{\partial t} = \frac{\partial C(K, t)}{\partial t} \frac{1}{P_d(0, t)} + f_d(0, t) \tilde{C}(t, K),
\]

\[
\frac{\partial^2 \tilde{C}(t, K)}{\partial K^2} = \frac{\partial^2 [\frac{C(K, t)}{P_d(0, t)}]}{\partial K^2} = \frac{1}{P_d(0, t)} \frac{\partial^2 C(t, K)}{\partial K^2}.
\]

Substituting these expressions into the last equation, we obtain the expression of the local volatility \( \sigma^2(t, K) \) in terms of call prices \( C(K, t) \)

\[
\sigma^2(t, K) = \frac{\frac{\partial C(K, t)}{\partial t} - P_d(0, t) E^Q_t [(r_d(t)K - r_f(t) S(t))] 1_{\{S(t) > K\}} }{\frac{1}{2} K^2 \frac{\partial^2 C(K, t)}{\partial K^2}}.
\]
Calibration
Before pricing any derivatives with a model, it is usual to calibrate it on the vanilla market,

determine all parameters present in the different stochastic processes which define the model in such a way that all European option prices derived in the model are as consistent as possible with the corresponding market ones.
The calibration procedure for the three-factor model with local volatility can be decomposed in three steps:

1. Parameters present in the Hull-White one-factor dynamics for the domestic and foreign interest rates, $\theta_d(t), \alpha_d(t), \sigma_d(t), \theta_f(t), \alpha_f(t), \sigma_f(t)$, are chosen to match European swaption / cap-floors values in their respective currencies.

2. The three correlation coefficients of the model, $\rho_{Sd}, \rho_{Sf}$ and $\rho_{df}$ are usually estimated from historical data.

3. After these two steps, the calibration problem consists in finding the local volatility function of the spot FX rate which is consistent with an implied volatility surface.
Calibration

\[
\sigma^2(t, K) = \frac{\partial C(K, t)}{\partial t} - P_d(0, t) \mathbb{E}^Q_t[(Kr_d(t) - r_f(t)S(t))1_{\{S(t) > K\}}] \frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}.
\]

Difficult because of \( \mathbb{E}^Q_t[(Kr_d(t) - r_f(t)S(t))1_{\{S(t) > K\}}] \):

- there exists no closed form solution
- it is not directly related to European call prices or other liquid products.
- Its calculation can obviously be done by using numerical methods but you have to solve (numerically) a three-dimensional PDE:

\[
0 = \frac{\partial \phi_F}{\partial t} + (r_d(t) - f_d(0, t)) \phi_F + \frac{\partial[(r_d(t) - r_f(t))S(t)\phi_F]}{\partial x} + \frac{\partial[(\theta_d(t) - \alpha_d(t) r_d(t))\phi_F]}{\partial y} + \frac{\partial[(\theta_f(t) - \alpha_f(t) r_f(t))\phi_F]}{\partial z} - \frac{\partial^2[\sigma^2(t, S(t))S(t)\phi_F]}{\partial x^2} - \frac{\partial^2[\sigma^2(t, S(t))\phi_F]}{\partial y^2} - \frac{\partial^2[\sigma^2(t, S(t))\phi_F]}{\partial z^2} - \frac{\partial^2[\sigma_d(t)\sigma_d(t)S_d(t)\phi_F]}{\partial x \partial y} - \frac{\partial^2[\sigma_d(t)\sigma_f(t)\rho_{Sd} \phi_F]}{\partial x \partial z} - \frac{\partial^2[\sigma_f(t)\rho_{df} \phi_F]}{\partial y \partial z}.
\]

(7)
Calibration : Comparison between local volatility with and without stochastic interest rates


- In a deterministic interest rates framework, the local volatility function $\sigma_{1f}(t, K)$ is given by the well-known Dupire formula:

$$\sigma_{1f}^2(t, K) = \frac{\partial C(K, t)}{\partial t} + K(f_d(0, t) - f_f(0, t)) \frac{\partial C(K, t)}{\partial K} + f_f(0, t) C(K, t) \cdot \frac{1}{2} K^2 \frac{\partial^2 C(K, t)}{\partial K^2}.$$ 

- If we consider the three-factor model with stochastic interest rates, the local volatility function is given by

$$\sigma_{3f}^2(t, K) = \frac{\partial C(K, t)}{\partial t} - P_d(0, t) E^Q_t [(Kr_d(t) - rf(t) S(t)) 1_{\{S(t) > K\}}] \cdot \frac{1}{2} K^2 \frac{\partial^2 C(K, t)}{\partial K^2}.$$ 

- We can derive the following interesting relation between the simple Dupire formula and its extension

$$\sigma_{3f}^2(t, K) - \sigma_{1f}^2(t, K) = KP_d(0, t) \{ \text{Cov}^Q_t [rf(t) - r_d(t), 1_{\{S(t) > K\}}] + \frac{1}{K} \text{Cov}^Q_t [rf(t), (S(t) - K)^+] \} \cdot \frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}.$$ 

(8)
Calibrating the local volatility by mimicking stochastic volatility models


If there exists a local volatility such that the one-dimensional probability distribution of the spot FX rate with the diffusion

$$dS(t) = (r_d(t) - r_f(t)) \ S(t) \ dt + \sigma(t, S(t)) \ S(t) \ dW^S_{DRN}(t),$$

is the same as the one of the spot FX rate with dynamics

$$dS(t) = (r_d(t) - r_f(t)) \ S(t) \ dt + \gamma(t, \nu(t)) \ S(t) \ dW^S_{DRN}(t)$$

for every time $t$ with $(\nu_t)_t$ a stochastic process, then this local volatility function has to satisfy

$$\sigma^2(t, K) = \frac{E^Q_d[\gamma^2(t, \nu(t)) e^{-\int_0^t r_d(s) ds} | S(t) = K]}{E^Q_d[e^{-\int_0^t r_d(s) ds} | S(t) = K]}$$

$$= E^Q_t[\gamma^2(t, \nu(t)) | S(t) = K].$$
References:
Pelsser (2003), Ballotta & Haberman (2003), Boyle & Hardy (2003) and local volatility papers
GAO definition:

- Consider at time 0 an $x_0 = x - T$ year old policyholder who receives at the age $x$ at time $T$ the payout of his capital policy which is linked to a fund with value $S(T)$ at $T$ (an index or of a portfolio).
- The policyholder has the right to choose at time $T$
  - either a cash payment equal to the investment in the equity fund $S(T)$
  - or an annual payment of $S(T)r^G_x$, with $r^G_x$ being a fixed rate called the Guaranteed Annuity rate

The cash payment equal to the investment in the equity fund $S(T)$ can imply and therefore can be considered as an annual payment of $S(T)r_x(T)$, with $r_x(T)$ being the market annuity payout rate over an initial single premium of 1, namely $r_x(T) = \frac{1}{\bar{a}_x(T)}$ and $\bar{a}_x(T) = \sum_{n=0}^{\omega-x} np_x P(T, T + n)$ with $np_x$ denoting the probability that the remaining lifetime of the policyholder is greater than $n$.

Therefore the guaranteed annuity contract with GAO can be seen as follows:
Introduction

At time $T$ the value $V(T)$ of the guaranteed annuity contract with GAO is given by

$$V(T) = S(T) \max(r_x^G, r_x(T)) \sum_{n=0}^{\omega-x} np_x P(T, T + n)$$

$$= S(T) + S(T) \max((r_x^G \sum_{n=0}^{\omega-x} np_x P(T, T + n)) - 1, 0)$$

since $r_x(T) = \frac{1}{\bar{a}_x(T)}$ and $\bar{a}_x(T) = \sum_{n=0}^{\omega-x} np_x P(T, T + n)$

Very popular in the 1970’s and 1980’s in the United Kingdom ($r_x^G = 11\%$).

Since the interest rate levels decreased in the 1990’s and the (expected) mortality rates increased, the value of the GAO's increased rapidly.

Similar options in US and Japan as part of variable annuity products: GMIB (Guaranteed Minimum Income Benefit).

Given the long maturities of these insurance products, volatility modelling is important.
Literature review

- Pricing GAO in a constant volatility framework (with stochastic interest rates):
  Dunbar (1999), Yang (2001), Wilkie et al. (2003), Boyle and Hardy (2003), Pelsser (2003), Ballotta and Haberman (2003), Chu and Kwok (2006), ....

- Pricing GAO in a stochastic volatility framework (with stochastic interest rates):

- Pricing GAO in a local volatility framework (with stochastic interest rates):
  ?

- Pricing GAO in a hybrid volatility framework (with stochastic interest rates):
  ?
The two-factor model with local volatility

The spot $S$ (capital policy) is governed by the following dynamics

$$dS(t) = (r(t) - q)S(t)dt + \sigma(t, S(t))S(t)dW^Q_S(t), \quad (9)$$

- $\sigma(t, S(t))$ is a local volatility function which allows to take into account the risk linked to the volatility of the spot.

Interest rates denoted by $r(t)$ follow a Hull-White one factor Gaussian model defined by the Ornstein-Uhlenbeck processes

$$dr(t) = [\theta(t) - \alpha(t)r(t)]dt + \sigma_r(t)dW^Q_r(t), \quad (10)$$

- $\theta(t), \sigma_r(t)$ and $\alpha(t)$ are deterministic functions of time in general. In our calibration we will take $\sigma_r$ and $\alpha$ constant.

Equations (9) and (10) are expressed in the (domestic) risk-neutral measure.

$W^Q_S(t)$ and $W^Q_r(t)$ are correlated Brownian motions

$$dW^Q_S(t)dW^Q_r(t) = \rho_{Sr}dt$$
The model

**Remark** The market dynamics could be better approximated by a hybrid volatility model that contains both stochastic volatility dynamics and local volatility ones.

**example:**

\[
\begin{align*}
    dS(t) &= (r(t) - q)S(t)dt + \sigma_{LOC2}(t, S(t))\sqrt{\nu(t)}S(t)dW^Q_S(t), \\
    dr(t) &= [\theta(t) - \alpha(t)r(t)]dt + \sigma_d(t)dW^Q_r(t), \\
    d\nu(t) &= \kappa(\theta - \nu(t))dt + \xi\sqrt{\nu(t)}dW^Q_\nu(t).
\end{align*}
\]

The local volatility function $\sigma_{LOC2}(t, S(t))$ can be calibrated from the local volatility that we have in a pure local volatility model.
Pricing methods
Pricing methods

At time $T$ the value $V(T)$ of the guaranteed annuity contract with GAO is given by

$$V(T) = S(T) \max(r^G_x, r_x(T)) \sum_{n=0}^{\omega-x} np_x P(T, T + n)$$

$$= S(T) + S(T) \max(((r^G_x \sum_{n=0}^{\omega-x} np_x P(T, T + n)) - 1, 0)$$

since $r_x(T) = \frac{1}{\bar{a}_x(T)}$ and $\bar{a}_x(T) = \sum_{n=0}^{\omega-x} np_x P(T, T + n)$

At time $t = 0$ the value of the guaranteed annuity contract with GAO (entered by an $x_0 = x - T$ year old policyholder) $V(x, 0, T)$ is given by

$$V(x, 0, T) = E_Q[e^{-\int_0^T r(s)ds}V(T) 1_{(\tau_{x_0}>T)}|\mathcal{F}_0]$$

$$= E_Q[e^{-\int_0^T r(s)ds}V(T)] E_Q[1_{(\tau_{x_0}>T)}]$$

where $\tau_{x_0}$ is a random variable which represents the remaining lifetime of the policyholder aged $x_0$, and where we assume that the mortality risk is unsystematic and independent of the financial risk.
Using

\[ E_Q[1_{\{\tau_{x_0} > T\}}] = \tau p_{x_0}, \]

\[
V(x, 0, T) = \tau p_{x_0} E_Q[\exp^{-\int_0^T r(s)ds} V(T)],
\]

\[
= \tau p_{x_0} \{ E_Q[\exp^{-\int_0^T r(s)ds} S(T)] + C(x, 0, T) \}
\]

(11)

\[
GAO = \tau p_{x_0} C(x, 0, T)
\]

\[
= \tau p_{x_0} E_Q[\exp^{-\int_0^T r(s)ds} S(T)] \max((r_x^G \sum_{n=0}^{\omega-x} n p_x P(T, T+n)) - 1, 0)]
\]

where \( n p_x \) is the probability that the remaining lifetime of the policyholder is strictly greater than \( n \)
Pricing methods

At time $t = 0$ the value of the GAO entered by an $x_0 = x - T$ year old policyholder is given by

$$
GAO = T p_{x_0} C(x, 0, T)
$$

$$
= T p_{x_0} E_Q[e^{-\int_0^T r(s)ds} S(T)\max((r_x^G \sum_{n=0}^{\infty} n p_x P(T, T + n)) - 1, 0)]
$$

By the density process $\xi_T = \frac{dQ_S}{dQ} |_{\mathcal{F}_T} = e^{-\int_0^T r(s)ds} \frac{S(T)}{S(0)}$ (where the numeraire is the equity price), we define a new probability measure $Q_S$ equivalent to the measure $Q$ such that (Geman et al., 1995)

$$
T p_{x_0} C(x, 0, T) = T p_{x_0} E_Q[e^{-\int_0^T r(s)ds} r_x^G S(T) (\sum_{n=0}^{\infty} n p_x P(T, T + n) - K)^+]
$$

$$
= T p_{x_0} r_x^G S(0) E_{Q_S}[(\sum_{n=0}^{\infty} n p_x P(T, T + n) - K)^+]
$$

with $K = \frac{1}{r_x^G}$. 

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Pricing methods

The model under $Q_S$, with $\sigma_r$ and $\alpha$ constant:

$$dS(t) = [r(t) - q + \sigma^2(t, S(t))]S(t)dt + \sigma(t, S(t))S(t)dW^{Q_S}_S(t)$$

$$dx(t) = [-\alpha x(t) + \rho_r \sigma_r \sigma(t, S(t))]dt + \sigma_r dW^{Q_S}_r(t)$$

where we have rewritten stochastic interest rates as a sum of a stochastic and a deterministic part (see e.g. Brigo and Mercurio):

$$r(t) = x(t) + \bar{x}(t), \quad (12)$$

where the deterministic part obeys the dynamics:

$$d\bar{x}(t) = (\theta(t) - \alpha \bar{x}(t))dt \quad (13)$$

and moreover (see e.g. Brigo and Mercurio)

$$\bar{x}(t) = f^{mkt}(0, t) + \frac{\sigma^2_r}{2\alpha^2}(1 - e^{-\alpha t})^2, \quad (14)$$

with $f^{mkt}(0, t)$ the market instantaneous forward rate at time 0 for the maturity $t$. 
Pricing methods

In this model, the time $T$ price of a zero-coupon bond $P(T, T + n)$ maturing at time $T + n$ is given by

$$P(T, T + n) = A(T, T + n)e^{-B(T, T+n)x(T)}$$

where

$$A(T, T + n) = \frac{P^{mkt}(0, T + n)}{P^{mkt}(0, T)} e^{-\frac{1}{2}[V(0, T+n) - V(0, T) - V(T, T+n)]}$$

$$B(T, T + n) = \frac{1}{\alpha}(1 - e^{-\alpha(T+n-T)})$$

$$V(u, v) = \frac{\sigma^2}{\alpha^2}[v - u + \frac{2}{\alpha}e^{-\alpha(v-u)} - \frac{1}{2\alpha}e^{-2\alpha(v-u)} - \frac{3}{2\alpha}]$$

with $P^{mkt}(0, T)$ denoting the market’s discounting factor maturing at $T$.

$$GAO = \tau p_x C(x, 0, T) = \tau p_x r_x^G S(0) E_Q S\left[\sum_{n=0}^{\omega-x} n p_x A(T, T + n) e^{-B(T, T+n)x(T)} - K\right]$$
Calibration
Before pricing any derivatives with a model, it is usual to calibrate it on the vanilla market,

determine all parameters present in the different stochastic processes which define the model in such a way that all European option prices derived in the model are as consistent as possible with the corresponding market ones.
The calibration procedure for the two-factor model with local volatility can be decomposed in three steps:

1. Parameters present in the Hull-White one-factor dynamics for the interest rates, \( \theta(t) \), \( \alpha \) and \( \sigma_r \), are chosen to match European swaption / cap-floors values.
2. The correlation coefficient of the model, \( \rho_{rs} \), is usually estimated from historical data.
3. After these two steps, the calibration problem consists in finding the local volatility function of the spot which is consistent with the implied volatility surface.
Calibration

\[ \sigma^2(T, K) = \frac{\partial C(K, T)}{\partial T} + qC(K, T) - qK \frac{\partial C(K, T)}{\partial K} + KP(0, T)E^Q_T [r(T)1_{\{S(T) > K\}}] \].

Difficult because of \( E^Q_T [r(T)1_{\{S(T) > K\}}] \):

- There exists no closed form solution
- It is not directly related to European call prices or other liquid products.
- Its calculation can obviously be done by numerical integration methods but you have to solve (numerically) a two-dimensional PDE:

\[
0 = \frac{\partial \phi_F}{\partial T} + (r(T) - f(0, T))\phi_F + \frac{\partial [(r(T) - q)S(T)\phi_F]}{\partial x} + \frac{\partial [(\theta(T) - \alpha(T)r(T))\phi_F]}{\partial y} - \frac{1}{2} \frac{\partial^2 [\sigma^2(T, S(T))S^2(T)\phi_F]}{\partial x^2} - \frac{1}{2} \frac{\partial^2 [\sigma^2(T)\phi_F]}{\partial y^2} - \frac{\partial^2 [\sigma(T, S(T))S(T)\sigma_r(T)\rho_{Sr}\phi_F]}{\partial x \partial y}
\]
Monte Carlo simulations

\[
\sigma^2(T, K) = \frac{\partial C(K, T)}{\partial T} + qC(K, T) - qK \frac{\partial C(K, T)}{\partial K} + KP(0, T) E^{Q_T}[r(T)1_{\{S(T) > K\}}] + \frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}.
\]

Monte Carlo techniques are possible for determining

\[E^{Q_T}[r(T)1_{\{S(T) > K\}}]\]

This expectation is expressed under the \(T\)-forward neutral measure \(Q_T\) and therefore we have to use the dynamics of \(S(t)\), \(r(t)\) and thus \(x(t)\) under that measure.

\[
\begin{align*}
    dS(t) &= (r(t) - q - \sigma(t, S(t))\sigma_r B(t, T)\rho_{Sr} S(t))dt + \sigma(t, S(t))S(t) dW_{S}^{TF}(t), \\
    dx(t) &= [-\alpha x(t) - \sigma_r^2 B(t, T)] dt + \sigma_r dW_{r}^{TF}(t),
\end{align*}
\] (15) (16)
The Euler discretisations of equations (15) and (16) are the following:

\begin{align}
S(t_{k+1}) &= S(t_k) + (r(t_k) - q - \sigma(t_k, S(t_k))) \sigma_r B(t_k, T) \rho_{Sr} S(t_k) \Delta t \\
&\quad + \sigma(t_k, S(t_k)) S(t_k) \sqrt{\Delta t} Z_S 
\end{align}

\begin{align}
x(t_{k+1}) &= x(t_k) + [-\alpha x(t_k) - \sigma_r^2 B(t_k, T)] \Delta t + \sigma_r \sqrt{\Delta t} [\rho_{Sr} Z_S + \sqrt{1 - \rho_{Sr}^2} Z_r] \\
r(t_{k+1}) &= x(t_{k+1}) + \bar{x}(t_{k+1})
\end{align}

where

\[ \bar{x}(t) = f^{mkt}(0, t) + \frac{\sigma_r^2}{2\alpha^2} (1 - e^{-\alpha t})^2, \]

and \( Z_r \) and \( Z_S \) are two independent standard normal variables.
First step: The goal is to determine the local volatility function at the first time step $T = T_1$ for all strike $K$.

In this step we assume that the local volatility is given by the local volatility associated to the deterministic interest rate case (Dupire). Knowing that function we can simulate $S(T_1)$ and $r(T_1)$ by using the Euler decompositions (18) and (19):

\begin{align*}
S(T_1) &= S(0) + (r(0) - q - \sigma(0, S(0))\sigma_r B(0, T)\rho_{Sr})S(0)T_1 + \sigma(0, S(0))S(0)\sqrt{T_1}Z_S \\
\bar{x}(T_1) &= x(0) + [\alpha x(0) - \sigma^2_r B(0, T)](T_1 - 0) + \sigma_r \sqrt{(T_1 - 0)}[\rho_{Sr} Z_S + \sqrt{1 - \rho^2_{Sr}} Z_r] \\
r(T_1) &= x(T_1) + \bar{x}(T_1)
\end{align*}

where

\[ \bar{x}(T_1) = f_{mkt}(0, T_1) + \frac{\sigma^2_r}{2\alpha^2}(1 - e^{-\alpha T_1})^2. \]
Then we can compute the expectation $E^{QT}[r(T_1)1_{\{S(T_1)>K\}}]$ for all $K$ by using:

$$E^{QT}[r(T_1)1_{\{S(T_1)>K\}}] = \frac{1}{n} \sum_{i=1}^{n} r_i(T_1)1_{\{S_i(T_1)>K\}}$$ \hspace{1cm} (24)$$

This allows us to get the local volatility expression $\sigma(T_1,K)$ at time $T_1$ for all strike $K$. 
Following the same procedure we can easily calibrate the local volatility at time \( T_2 \) by using the local volatility obtained at time \( T_1 \). More precisely,

\[
S(T_2) = S(T_1) + (r(T_1) - q - \sigma(T_1, S(T_1))\sigma_r B(T_1, T)\rho_{Sr})S(T_1)(T_2 - T_1) \\
+ \sigma(T_1, S(T_1))S(T_1)\sqrt{(T_2 - T_1)}Z_S
\] (25)

\[
x(T_2) = x(T_1) + [-\alpha x(T_1) - \sigma_r^2 B(T_1, T)](T_2 - T_1) + \sigma_r \sqrt{(T_2 - T_1)}[\rho_{Sr}Z_S + \sqrt{1 - \rho_{Sr}^2}Z_r]
\] (26)

where

\[
\bar{x}(T_2) = f^{mkt}(0, T_2) + \frac{\sigma_r^2}{2\alpha^2}(1 - e^{-\alpha T_2})^2.
\] (27)
Then we can compute the expectation $\mathbb{E}^{Q_T}[r(T_2)1_{\{S(T_2) > K\}}]$ for all $K$ by using:

\[
\mathbb{E}^{Q_T}[r(T_2)1_{\{S(T_2) > K\}}] = \frac{1}{n} \sum_{i=1}^{n} r_i(T_2)1_{\{S_i(T_2) > K\}}
\] (28)

This allows us to get the local volatility expression $\sigma(T_2, K)$ at time $T_2$ for all strike $K$.

Between time $T_1$ and $T_2$, we can use an interpolation method (bicubic splines, ...).

Following this procedure we can generate the local volatility expression up to time $T = T_k$. 
Second method: by adjusting the Dupire formula
Calibration: Comparison between local volatility with and without stochastic interest rates

- In a deterministic interest rates framework, the local volatility function $\sigma_{1f}(T,K)$ is given by the well-known Dupire formula:

$$\sigma_{1f}^2(T,K) = \frac{\partial C(K,T)}{\partial T} + K(f(0,T) - q)\frac{\partial C(K,T)}{\partial K} + qC(K,T) \frac{1}{2} K^2 \frac{\partial^2 C(K,T)}{\partial K^2}.$$  

- If we consider the two-factor model with stochastic interest rates, the local volatility function is given by

$$\sigma_{2f}^2(T,K) = \frac{\partial C(K,T)}{\partial T} + qC(K,T) - qK\frac{\partial C(K,T)}{\partial K} - KP(0,T)\mathbb{E}_Q^T[r(T)\mathbb{1}_{\{S(T) > K\}}] \frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}.$$  

- We can derive the following interesting relation between the simple Dupire formula and its extension

$$\sigma_{2f}^2(T,K) - \sigma_{1f}^2(T,K) = \frac{P(0,T)\text{Cov}_Q^T[r(T),\mathbb{1}_{\{S(T) > K\}}]}{\frac{1}{2} K \frac{\partial^2 C}{\partial K^2}}.$$  

(29)
Third method: by mimicking stochastic volatility models
Calibrating the local volatility by mimicking stochastic volatility models

- Consider the following risk-neutral dynamics for the spot

\[ dS(t) = (r(t) - q)S(t)dt + \gamma(t, \nu(t))S(t)dW^Q_S(t) \]

- \( \nu(t) \) is a stochastic process which provides the stochastic perturbation for the spot volatility.
- Common choices:
  1. \( \gamma(t, \nu(t)) = \nu(t) \)
  2. \( \gamma(t, \nu(t)) = \exp(\sqrt{\nu(t)}) \)
  3. \( \gamma(t, \nu(t)) = \sqrt{\nu(t)} \)

- The stochastic process \( \nu(t) \) is generally modelled by
  - a Cox-Ingersoll-Ross (CIR) process as for example the Heston stochastic volatility model:
    \[ d\nu(t) = \kappa(\theta - \nu(t))dt + \xi\sqrt{\nu(t)}dW^Q_\nu(t) \]
  - a Ornstein-Uhlenbeck process (OU) as for example the Schöbel and Zhu stochastic volatility model:
    \[ d\nu(t) = k[\lambda - \nu(t)]dt + \xi dW^Q_\nu(t) \]
Calibrating the local volatility by mimicking stochastic volatility models

Applying Tanaka’s formula to the non-differentiable function 
\[ e^{-\int_0^t r(s)ds} (S(t) - K)^+ , \]
where 
\[ dS(t) = (r(t) - q)S(t)dt + \gamma(t, \nu(t))S(t)dW^Q_S(t) \]
we obtain

\[
\frac{E^Q[\gamma^2(t, \nu(t))e^{-\int_0^t r(s)ds} | S(t) = K]}{E^Q[e^{-\int_0^t r(s)ds} | S(t) = K]} = \frac{\frac{\partial C(K, T)}{\partial t} + qC(K, t) - qK \frac{\partial C(K, t)}{\partial K} - KP(0, t)E^Q_t[r(t)1\{S(t) > K\}]}{\frac{1}{2}K^2 \frac{\partial^2 C}{\partial K^2}}
\]

Therefore, if there exists a local volatility such that the one-dimensional probability distribution of the spot with the diffusion

\[ dS(t) = (r(t) - q)S(t)dt + \sigma(t, S(t))S(t)dW^Q_S(t), \]
is the same as the one of the spot with dynamics

\[ dS(t) = (r(t) - q)S(t)dt + \gamma(t, \nu(t))S(t)dW^Q_S(t) \]
for every time \( t \), then this local volatility function has to satisfy

\[ \sigma^2(t, K) = \frac{E^Q[\gamma^2(t, \nu(t))e^{-\int_0^t r(s)ds} | S(t) = K]}{E^Q[e^{-\int_0^t r(s)ds} | S(t) = K]} = E^{Q_t}[\gamma^2(t, \nu(t)) | S(t) = K]. \]
A particular case with closed form solution

Consider the two-factor model with local volatility (LVHW)

\[
\begin{align*}
\quad \quad dS(t) &= (r(t) - q)S(t)dt + \sigma(t, S(t))S(t)dW_S^Q(t), \\
\quad dr(t) &= [\theta(t) - \alpha r(t)]dt + \sigma_r dW_r^Q(t),
\end{align*}
\]

Calibration by mimicking a Schöbel and Zhu / Hull and White stochastic volatility model (SZHW)

\[
\begin{align*}
\quad \quad dS(t) &= (r(t) - q)S(t)dt + \nu(t)S(t)dW_S(t) ,  \\
\quad dr(t) &= [\theta(t) - \alpha r(t)]dt + \sigma_r dW_r(t),  \\
\quad d\nu(t) &= k[\lambda - \nu(t)] dt + \xi dW_\nu(t),
\end{align*}
\]

The local volatility function is given by:

\[
\sigma^2(T, K) = E^Q_T[\nu^2(T)|S(T) = K] = E^Q_T[\nu^2(T)] \quad \text{if we assume independence between } S \text{ and } \nu = (E^Q_T[\nu(T)])^2 + \text{Var}^Q_T[\nu(T)]
\]
A particular case with closed form solution

- Under the $T$-Forward measure:

$$d\nu(t) = [k(\lambda - \nu(t)) - \rho_r\nu\sigma_r b(t, T)\xi]dt + \xi\ dW^{TF}_\nu(t)$$

$$\nu(T) = \nu(t)e^{-k(T-t)} + \int_t^T k(\lambda - \frac{\rho_r\nu\sigma_r b(u, T)\xi}{k})e^{-k(T-u)}du + \int_t^T \xi e^{-k(T-t)}dW^{TF}_\nu(u)$$

where $b(t, T) = \frac{1}{\alpha}(1 - e^{-\alpha(T-t)})$

- so that $\nu(T)$ conditional on $\mathcal{F}_t$ is normally distributed with mean and variance given respectively by

$$E^{Q_T}[\nu(T)|\mathcal{F}_t] = \nu(t)e^{-k(T-t)} + (\lambda - \frac{\rho_r\nu\sigma_r\xi}{\alpha k})(1 - e^{-k(T-t)})$$

$$+ \frac{\rho_r\nu\sigma_r\xi}{\alpha(\alpha + k)}(1 - e^{-(\alpha+k)(T-t)})$$

$$\text{Var}^{Q_T}[\nu(T)|\mathcal{F}_t] = \frac{\xi^2}{2k}(1 - e^{-2k(T-t)})$$
A particular case with closed form solution

\[ \sigma^2(T, K) = (\mathbb{E}^{Q_T}[\nu(T)])^2 + \text{Var}^{Q_T}[\nu(T)] \]

\[ = \left( \nu(0)e^{-kT} + (\lambda - \frac{\rho_{r\nu}\sigma r\xi}{\alpha k})(1 - e^{-kT}) + \frac{\rho_{r\nu}\sigma r\xi}{\alpha(\alpha + k)}(1 - e^{-(\alpha + k)T}) \right)^2 \]

\[ + \frac{\xi^2}{2k}(1 - e^{-2kT}) \]

\[ = \sigma^2(T) \]

**Figure:** \( \xi = 20\%, \ k = 50\%, \ \alpha = 5\%, \ \nu(0) = 10\%, \ \sigma = 1\%, \ \lambda = 20\%, \ \rho_{r\nu} = 1\% \)
Numerical results
We have first calibrated the local volatility and the interest rate parameters to market option data per end July 2007 given in the paper A. van Haastrecht, R. Plat and A. Pelsser, Valuation of Guaranteed Annuity Options Using a Stochastic Volatility Model for Equity Prices, IME, 2010, 47(3), 266-277.

More precisely, a calibration of the Hull and White interest rate model to the U.S. swaptions market leads to $\alpha = 0.025773776$ and $\sigma_r = 0.009578297$. 
The equity considered in this section is the S&P500 (U.S.). van Haastreicht et al. determined the effective (10 years) correlation between the stock and the interest rates in the BSHW process by using time series analysis of the interest rates and S&P500 (U.S.) index over the period from February 2002 to July 2007. They found a correlation coefficient $\rho_{Sr} = 14.64\%$ between the interest rates and the equity price. We took $q = 0$. 
As van Haastrecht et al. considered GAO with a 10 year maturity, they calibrated the equity to the terminal distributions of the equity price at that time.

In our case the calibration of the volatility of the equity consists in building the local volatility surface given by the equation

\[ \sigma^2(T, K) = \frac{\partial C(K, T)}{\partial T} + qC(K, T) - qK \frac{\partial C(K, T)}{\partial K} + KP(0, T)E_Q[ r(T) 1_{\{S(T) > K\}} ] + \frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}. \]

We need the local volatility for all strikes and all maturities. As a first simple case, we have assumed that the implied volatility is constant with respect to the maturity. A plot of the complete implied volatility surface and the resulting market call option prices can be found on the next slides. Following the Monte Carlo approach, we have found the corresponding local volatility surface.
Figure: Implied volatility surface
Figure: The resulting market call option prices
In the table on the next slide, we compare the implied volatility (for a range of seven different strikes and a fixed maturity $T = 10$) with volatility given by all the three models after being calibrated. We can see that the local volatility model (LVHW) and the stochastic volatility model (SZHW) are both well calibrated since they are able to generate the Smile/Skew quite close to the implied one. However, contrarily to the SZHW model, the LVHW model is calibrated over all maturities. This will have an influence upon the hedging strategy. A plot of the Smile/Skew generated by the market and these three models is given in a figure.
**Numerical results**

<table>
<thead>
<tr>
<th>Strike</th>
<th>Mkt vol</th>
<th>SZHW vol</th>
<th>LVHW vol</th>
<th>BSHW (ATM vol)</th>
</tr>
</thead>
<tbody>
<tr>
<td>80.00</td>
<td>27.50%</td>
<td>27.50%</td>
<td>27.56%</td>
<td>25.80%</td>
</tr>
<tr>
<td>90.00</td>
<td>26.60%</td>
<td>26.60%</td>
<td>26.55%</td>
<td>25.80%</td>
</tr>
<tr>
<td>95.00</td>
<td>26.20%</td>
<td>26.20%</td>
<td>26.29%</td>
<td>25.80%</td>
</tr>
<tr>
<td>100.00</td>
<td>25.80%</td>
<td>25.80%</td>
<td>25.71%</td>
<td>25.80%</td>
</tr>
<tr>
<td>105.00</td>
<td>25.40%</td>
<td>25.40%</td>
<td>25.46%</td>
<td>25.80%</td>
</tr>
<tr>
<td>110.00</td>
<td>25.00%</td>
<td>25.00%</td>
<td>25.10%</td>
<td>25.80%</td>
</tr>
<tr>
<td>120.00</td>
<td>24.30%</td>
<td>24.40%</td>
<td>24.61%</td>
<td>25.80%</td>
</tr>
</tbody>
</table>

**Table**: calibration results
Figure: Smile/Skew generated by the market and the three models after calibration
Now we will compare prices of GAOs obtained with our LVHW model to prices given by using the BSHW and the SZHW model computed in van Haastrecht et al..
We consider, as in van Haastrecht et al., that the policyholder is 55 years old, the retirement age is 65, leading to a maturity $T$ of the GAO option of 10 years.
The fund value at time 0, $S(0) = 100$.
The survival rates are based on the PNMA00 table of the Continuous Mortality Investigation (CMI) for male pensioners (available at http://www.actuaries.org.uk/)
In the following table we show prices for the GAO given for different guaranteed annuity rates $r_x^G$ for each of the considered models. The results for the SZHW model are obtained using the closed form expression derived in van Haastrecht et al. and the pricing formula for the BSHW is derived in Ballotta and Haberman (2003). The results for the LVHW model are obtained by using Monte Carlo simulations (50 000 simulations and 500 steps). The following table and figure show the corrections induced by the LVHW and the SZHW models with respect to the BSHW model.
<table>
<thead>
<tr>
<th>$r^G_x$</th>
<th>BSHW</th>
<th>SZHW</th>
<th>LVHW</th>
<th>stderr</th>
<th>$\pm$ 95 % interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>7%</td>
<td>0.88</td>
<td>1.04</td>
<td>0.971685</td>
<td>0.00749</td>
<td>0.014681</td>
</tr>
<tr>
<td>8%</td>
<td>3.11</td>
<td>3.54</td>
<td>3.3496</td>
<td>0.01369</td>
<td>0.026833</td>
</tr>
<tr>
<td>8.44%</td>
<td>4.84</td>
<td>5.43</td>
<td>5.11786</td>
<td>0.016584</td>
<td>0.032505</td>
</tr>
<tr>
<td>9%</td>
<td>7.74</td>
<td>8.53</td>
<td>8.09533</td>
<td>0.019977</td>
<td>0.039155</td>
</tr>
<tr>
<td>10%</td>
<td>14.9</td>
<td>16.06</td>
<td>15.5536</td>
<td>0.025559</td>
<td>0.050095</td>
</tr>
<tr>
<td>11%</td>
<td>23.96</td>
<td>25.42</td>
<td>24.7955</td>
<td>0.02916</td>
<td>0.057153</td>
</tr>
<tr>
<td>12%</td>
<td>34.06</td>
<td>35.73</td>
<td>35.0336</td>
<td>0.031747</td>
<td>0.062223</td>
</tr>
<tr>
<td>13%</td>
<td>44.58</td>
<td>46.43</td>
<td>45.7426</td>
<td>0.033302</td>
<td>0.065272</td>
</tr>
</tbody>
</table>

**Table:** GAO total value
Figure: LVHW and the SZHW models corrections with respect to the BSHW model
<table>
<thead>
<tr>
<th>$r^G_x$</th>
<th>SZHW-BSHW</th>
<th>LVHW-BSHW</th>
<th>LVHWinf-BSHW</th>
<th>LVHWsup-BSHW</th>
</tr>
</thead>
<tbody>
<tr>
<td>7%</td>
<td>0.16</td>
<td>0.0916854</td>
<td>0.07700415</td>
<td>0.10636585</td>
</tr>
<tr>
<td>8%</td>
<td>0.43</td>
<td>0.2396</td>
<td>0.21276701</td>
<td>0.26643299</td>
</tr>
<tr>
<td>8.44%</td>
<td>0.59</td>
<td>0.27786</td>
<td>0.24535536</td>
<td>0.31036464</td>
</tr>
<tr>
<td>9%</td>
<td>0.79</td>
<td>0.35533</td>
<td>0.31617528</td>
<td>0.39448472</td>
</tr>
<tr>
<td>10%</td>
<td>1.16</td>
<td>0.6536</td>
<td>0.60350514</td>
<td>0.70369486</td>
</tr>
<tr>
<td>11%</td>
<td>1.46</td>
<td>0.8355</td>
<td>0.77834718</td>
<td>0.89265282</td>
</tr>
<tr>
<td>12%</td>
<td>1.67</td>
<td>0.9736</td>
<td>0.91137666</td>
<td>1.03582334</td>
</tr>
<tr>
<td>13%</td>
<td>1.85</td>
<td>1.1626</td>
<td>1.09732828</td>
<td>1.22787172</td>
</tr>
</tbody>
</table>

**Table:** Corrections
The following table presents the time value given by the difference between the option value and its intrinsic value. The volatility of an option is an important factor for this time value since this value depends on the time until maturity and the volatility of the underlying instrument’s price. The following figure is a plot of the time value given by all considered models.
<table>
<thead>
<tr>
<th>$r^G_x$</th>
<th>BSHW</th>
<th>SZHW</th>
<th>LVHW</th>
</tr>
</thead>
<tbody>
<tr>
<td>7%</td>
<td>0.88</td>
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<td>8.44%</td>
<td>4.84</td>
<td>5.43</td>
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<td>6.47</td>
<td>7.27</td>
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<td>3.6436</td>
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<tr>
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<td>1.4</td>
<td>2.86</td>
<td>2.2355</td>
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<tr>
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<td>0.86</td>
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<td>1.8336</td>
</tr>
<tr>
<td>13%</td>
<td>0.74</td>
<td>2.58</td>
<td>1.9026</td>
</tr>
</tbody>
</table>

**Table:** GAO time value
Numerical results

Guaranteed Annuity Options

Figure: Time value

Griselda.Deelstra@ulb.ac.be (ULB)
Future research
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- sensibility studies, other market studies, ...
- hedging problem by using Likelihood ratio method
- hedging in local volatility versus stochastic volatility model
- influence for path-dependent insurance products like Ratchets
Thank you for your attention