Risk Measurement in Credit Portfolio Models

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Quantitative Risk Management

- Profit & loss distributions (P & L) are very complex.

  Appropriate summary statistics are needed:

  - Standardization facilitates communication
  - Simple tools for sensitivity analysis
  - Basis for capital regulation of financial firms

"Capital regulation is the cornerstone of bank regulators' efforts to maintain a safe and sound banking system, a critical element of overall financial stability."

(Ben S. Bernanke, 2006)

This requires appropriate ways to measure the downside risk of financial positions.

Bailed out by the Tax Payer

Outline

(i) Review of risk measures

- The industry standard: Value at risk
- Convex risk measures
- An example: Utility-based Shortfall Risk

(ii) Implementation for credit portfolios

- Monte Carlo Methods
- Exponential Twisting
- Stochastic Approximation

Value at Risk

Value at Risk in the Media

“David Einhorn, who founded Greenlight Capital, a prominent hedge fund, wrote not long ago that VaR was

> '*relatively useless as a risk-management tool and potentially catastrophic when its use creates a false sense of security among senior managers and watchdogs. This is like an air bag that works all the time, except when you have a car accident.*' ”

“Nicholas Taleb, the best-selling author of 'The Black Swan,' has crusaded against VaR for more than a decade. He calls it, flatly, '*a fraud.*' ”

(“Risk Mismanagement”, New York Times, 2. Januar 2009)

The Industry Standard – Value at Risk

Value at risk at level λ:

$$\mathrm{VaR}_\lambda(X) = \inf\{m \in \mathbb{R} : \ P[m + X < 0] \leq \lambda\}$$

"Smallest monetary amount to be added to a financial position such that the probability of a loss becomes smaller than λ."

Drawbacks of Value at Risk

- does not account for the size of extremely large losses
- does not encourage diversification

This motivates an axiomatic analysis of risk measures.

Value at Risk – Diversification

$$X_i = \begin{cases} 1 & \text{with probability } 50\% \\ -1 & \text{with probability } 50\% \end{cases}$$

The Value at Risk of X_i at level 50% is -1.

If X_1 and X_2 are stochastically independent, then

$$\frac{X_1 + X_2}{2} = \begin{cases} 1 & \text{with probability } 25\% \\ 0 & \text{with probability } 50\% \\ -1 & \text{with probability } 25\% \end{cases}$$

The VaR at level 50% of the diversified position is 0.

Value at Risk – Large Losses

$$X_1 = \begin{cases} 1 & \text{with probability } 99\% \\ -1 & \text{with probability } 1\% \end{cases}$$

$$X_2 = \begin{cases} 1 & \text{with probability } 99\% \\ -10^{10} & \text{with probability } 1\% \end{cases}$$

The VaR of both positions at level 1% is -1.

Axiomatic Theory

Static Risk Measures

Risk measures

$\rho : \mathcal{X} \to \mathbb{R}$

- Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- Cash invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.

Capital requirement

- A position $X \in \mathcal{X}$ is acceptable, if $\rho(X) \leq 0$.
  The collection $\mathcal{A}$ of all acceptable positions is the *acceptance set*.
- ρ is a capital requirement, i.e.

$$\rho(X) = \inf \{m \in \mathbb{R} : \; X + m \in \mathcal{A}\} .$$

Diversification

Semiconvexity:

$$\rho(\alpha X + (1-\alpha)Y) \leq \max(\rho(X), \rho(Y)) \qquad (\alpha \in [0,1]).$$

$\Longrightarrow$

Convexity (Föllmer & Schied, 2002):

$$\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y) \qquad (\alpha \in [0,1]).$$

Geometric properties of the acceptance set

- ρ convex $\Leftrightarrow$ $\mathcal{A}$ convex.

A Better Risk Measure: UBSR

Utility-based Shortfall Risk

$\ell : \mathbb{R} \to \mathbb{R}$ convex loss function, z interior point of the range of ℓ.

The acceptance set is defined as

$$\mathcal{A} = \{X \in L^\infty : \ E_P\left[\ell(-X)\right] \le z\}$$

$\mathcal{A}$ induces a convex risk measure ρ:

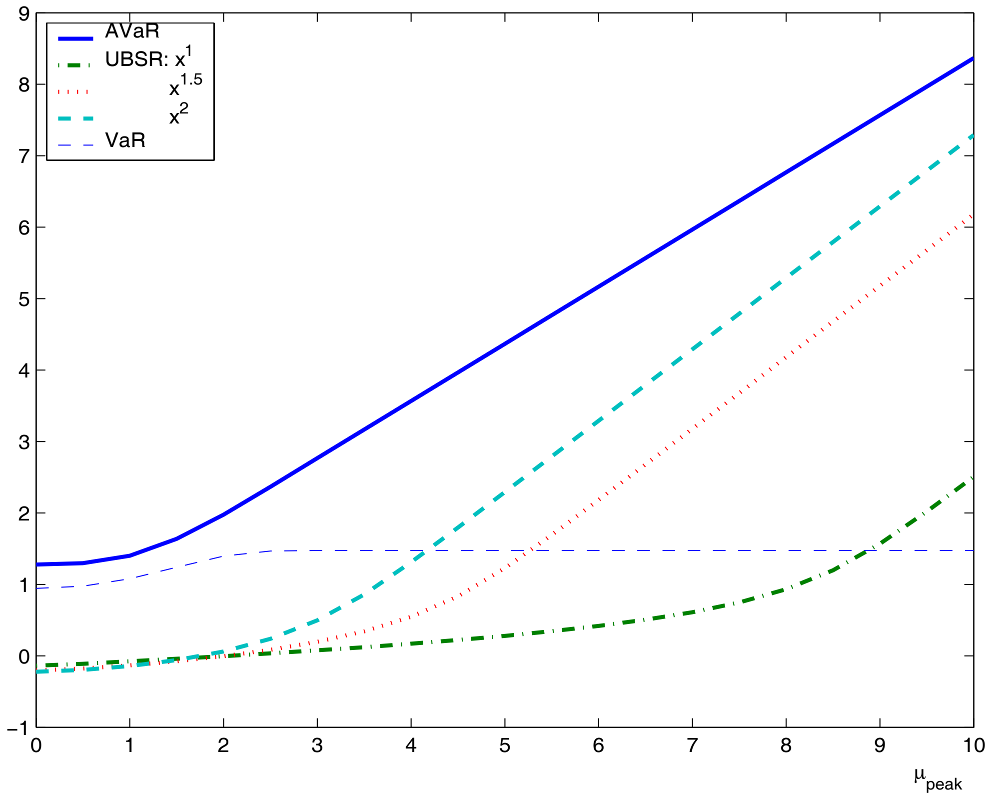
$$\rho(X) = \inf\{m \in \mathbb{R} : \ X + m \in \mathcal{A}\}$$

Simple formula

Shortfall risk $\rho(X)$ is given by the unique root s_* of the function

$$f(s) := E[\ell(-X - s)] - z.$$

Example: UBSR (2)



$\mathrm{VaR}_{0.05}$, $\mathrm{AVaR}_{0.05}$ und Utility-based Shortfall Risk with $p \in \{1, \frac{3}{2}, 2\}$ and $z = 0.3$ as functions of μ for a mixture of a Student t (weight 0.96) and a Gaussian with mean μ (weight 0.04).

Characterization Theorem

Shortfall risk has convex acceptance and rejection set on the level of distributions; on L^∞ the only one with this property.

Distribution-based risk measures

- Let $\mathcal{M}_{1,c}(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$.
- Distribution-based risk measure can be interpreted as functionals on $\mathcal{M}_{1,c}(\mathbb{R})$.

Acceptance and rejection set

- An acceptance set on the level of probability distributions can be defined by

$$\mathcal{N}_\rho = \{\mu \in \mathcal{M}_{1,c}(\mathbb{R}) : \ \rho(\mu) \leq 0\}.$$

Characterization Theorem (cont.)

Theorem 1 (W., 2006) *Let ρ be a distribution-based risk measure.*

Assume there exists $x \in \mathbb{R}$ with $\delta_x \in \mathcal{N}$ such that for $y \in \mathbb{R}$, $\delta_y \in \mathcal{N}^c$,

$$(1-\alpha)\delta_x + \alpha\delta_y \in \mathcal{N}$$

for sufficiently small $\alpha > 0$.

Then the following statements are equivalent:

(i) *$\mathcal{N}$ is ϕ-weakly closed for some gauge function $\phi : \mathbb{R} \to [1, \infty)$, $\mathcal{N}$ and $\mathcal{N}^c$ are both convex.*

(ii) *For some left-continuous loss function $\ell : \mathbb{R} \to \mathbb{R}$ and a scalar $z \in \mathbb{R}$ in the interior of the convex hull of the range of ℓ:*

$$\mathcal{N} = \left\{ \mu \in \mathcal{D} : \int \ell(-x)\mu(dx) \le z \right\}.$$

Implementation in Credit Risk Models

Better Manage Your Risks!

Credit Portfolios

- One-period model with time periods $t = 0, 1$
- Financial positions at $t = 1$ are modeled as random variables

Credit Portfolio Losses

- Portfolio with m positions (obligors)
- The random loss at time 1 due to a default of obligor $i = 1, 2, \ldots, m$ is denoted by l_i
- The total losses are given by

$$L = \sum_{i=1}^{m} l_i$$

- Typical decomposition: $l_i = v_i D_i$ with exposure v_i and default indicator $D_i \in \{0, 1\}$

Credit Portfolios (2)

- Framework above is completely general (if we focus on one-period credit loss models)
- Risk assessment in practice requires specific models that need to be estimated/calibrated and evaluated
- **Examples**
  - Credit Metrics
    - JP Morgan; based on Normal Copula
  - Credit Risk+
    - Credit Suisse; Poisson mixture model
  - Copula models like the t-copula model
    - general family; Gaussian mixture like t-copula particularly tractable

Credit Metrics

- JP Morgan's Credit Metrics is a simplistic toy model.
  - The dependence structure is based on a Gaussian copula and ad hoc.
  - The model exhibits no tail-dependence.
  - Credit Metrics model is "like Black-Scholes".
- Credit Metrics is also called Normal Copula Model (NCM).
- The NCM can be used as a basis for Gaussian mixture models like the t-copula model.
- Risk estimation techniques that work in the NCM can often be extended to Gaussian mixture models and other models.

Monte Carlo Simulation

Shortfall risk $\rho(X)$ with $L = -X$ is given by the unique root s_* of the function

$$f(s) := E[\ell(-X - s)] - z.$$

Efficient Computation

- Variance reduction techniques increase the accuracy/rate of convergence, e.g. importance sampling (Dunkel & W., 2007)
- Stochastic approximation (Dunkel & W., 2009)

Normal Copula Model

- Model of overall losses of credit portfolio over fixed time horizon
- Losses $L = -X \geq 0$ are given by:

$$L = \sum_{i=1}^{m} v_i D_i.$$

  - Default indicators:

$$D_i = \mathbf{1}_{\{Y_i > y_i\}}$$

  - Marginal default probabilities:

$$p_i = P\{D_i = 1\}$$

  - m-dimensional normal factor with standardized marginals:

$$Y = (Y_1, Y_2, \ldots, Y_m)$$

  - Threshold levels:

$$y_i = \Phi^{-1}(1 - p_i)$$

Normal Copula Model (continued)

In industry applications the covariance matrix of the Gaussian vector Y is often specified through a factor model:

$$
\begin{aligned}
Y_i &= A_{i0}\varepsilon_i + \sum_{j=1}^{d} A_{ij} Z_j \qquad i = 1, \ldots, m,\ d < m; \\
1 &= A_{i0}^2 + A_{i1}^2 + \ldots + A_{id}^2 \qquad A_{i0} > 0,\ A_{ij} \geq 0,
\end{aligned}
$$

where

- $Z_1, \ldots, Z_d$ are d independent standard normal random variables (systematic risks), and
- $\varepsilon_1, \ldots, \varepsilon_m$ are m independent standard normal random variables which are independent of $Z_1, \ldots, Z_d$ (idiosyncratic risks).

Importance Sampling

First task:

Estimate

$$E_P[\ell(L-s)] = E_P[h(L)]$$

with $h(L) = \ell(L-s)$.

Two-step variance reduction:

(i) Importance sampling for L conditional on factor Z.

(ii) Variance reduction for factor Z.

Special case: independent default events

In the factor model, independence corresponds to

$$A_{i0} = 1, \quad A_{ij} = 0 \qquad i = 1, \ldots, m, \quad j = 1, \ldots, d.$$

Importance Sampling:

- If Q is an equivalent probability measure with a density of the form

  $$\frac{dQ}{dP} = g(L),$$

  then $E_P[h(L)] = E_Q\left[\frac{h(L)}{g(L)}\right]$.

- Sampling L_k independently from the distribution of L under Q, we get an unbiased, consistent estimator of $E_P[h(L)]$:

  $$J_n^g = \frac{1}{n}\sum_{k=1}^{n} \frac{h(L_k)}{g(L_k)}.$$

Exponential twisting

SR loss function:

Suppose that $\ell(x) = \gamma^{-1} x^{\gamma} \mathbf{1}_{[0,\infty)}(x)$ is polynomial.

Measure change:

Consider class of probability measures Q_θ, $\theta \geq 0$, with

$$\frac{dQ_\theta}{dP} = \frac{\exp(\theta L)}{\psi(\theta)},$$

where $\psi(\theta) = \log E[\exp(\theta L)] = \sum_{i=1}^{m} \log[1 + p_i(e^{\theta v_i} - 1)]$.

'Optimal' measure change:

Minimize an upper bound for the L^2-error of the estimator of $E_P[\ell(L - s)]$. This suggests an 'optimal' θ_s.

Exponential twisting (continued)

(i) Calculate

$$q_i(\theta_s) := \frac{p_i e^{v_i \theta_s}}{1 + p_i \, (e^{v_i \theta_s} - 1)}.$$

(ii) Generate m Bernoulli-random numbers $D_i \in \{0, 1\}$, such that $D_i = 1$ with probability $q_i(\theta_s)$.

(iii) Calculate

$$\psi(\theta_s) = \sum_{i=1}^{m} \log[1 + p_i(e^{\theta_s v_i} - 1)]$$

and $L = \sum_{i=1}^{m} v_i D_i$, and return the estimator

$$\ell(L - s) \exp\left[-L\theta_s + \psi\left(\theta_s\right)\right].$$

Exponential twisting (continued)

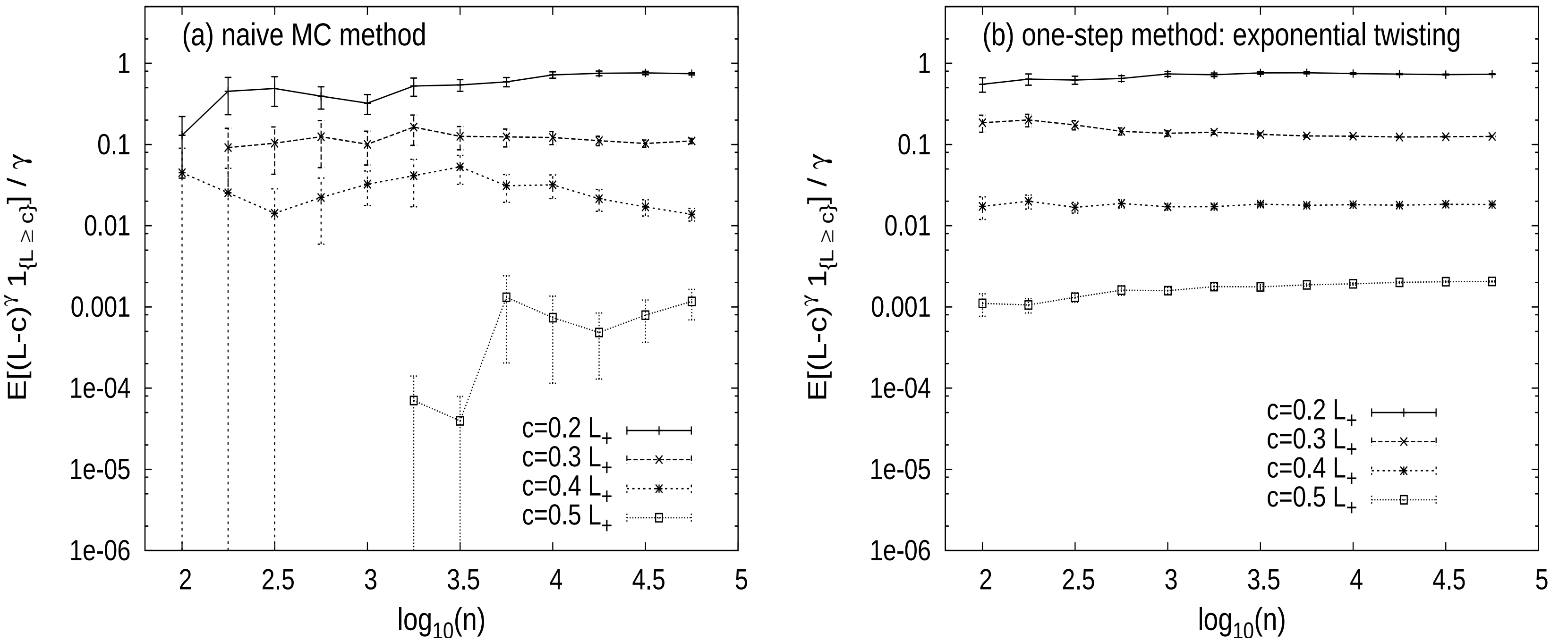


Figure 1: MC results for estimating SR with piecewise polynomial loss function in the NCM. Length of error bars is sample standard deviation of estimator.

Stochastic Approximation

Stochastic approximation methods provide more efficient root-finding techniques for shortfall risk (Dunkel & W., 2009).

Robbins-Monro Algorithm

- Let $\hat{Y}_s : [0,1] \to \mathbb{R}$ such that $E[\hat{Y}_s(U)] = f(s)$ for U ∼unif[0,1].
- Choose a constant $\gamma \in \left(\frac{1}{2}, 1\right]$, $c > 0$, and a starting value $s_1 \in [a,b] \ni s^*$.
- For $n \in \mathbb{N}$ we define recursively:

$$s_{n+1} \;=\; \Pi_{[a,b]}\left[s_n + \frac{c}{n^\gamma} \cdot Y_n\right] \tag{1}$$

with

$$Y_n = \hat{Y}_{s_n}(U_n) \tag{2}$$

for a sequence (U_n) of independent, unif[0,1]-distributed random variables.

Stochastic Approximation (2)

Averaging procedure

Theorem 2 *Suppose that* $\gamma \in \left(\frac{1}{2}, 1\right)$. *For arbitrary* $\rho \in (0,1)$ *we define*

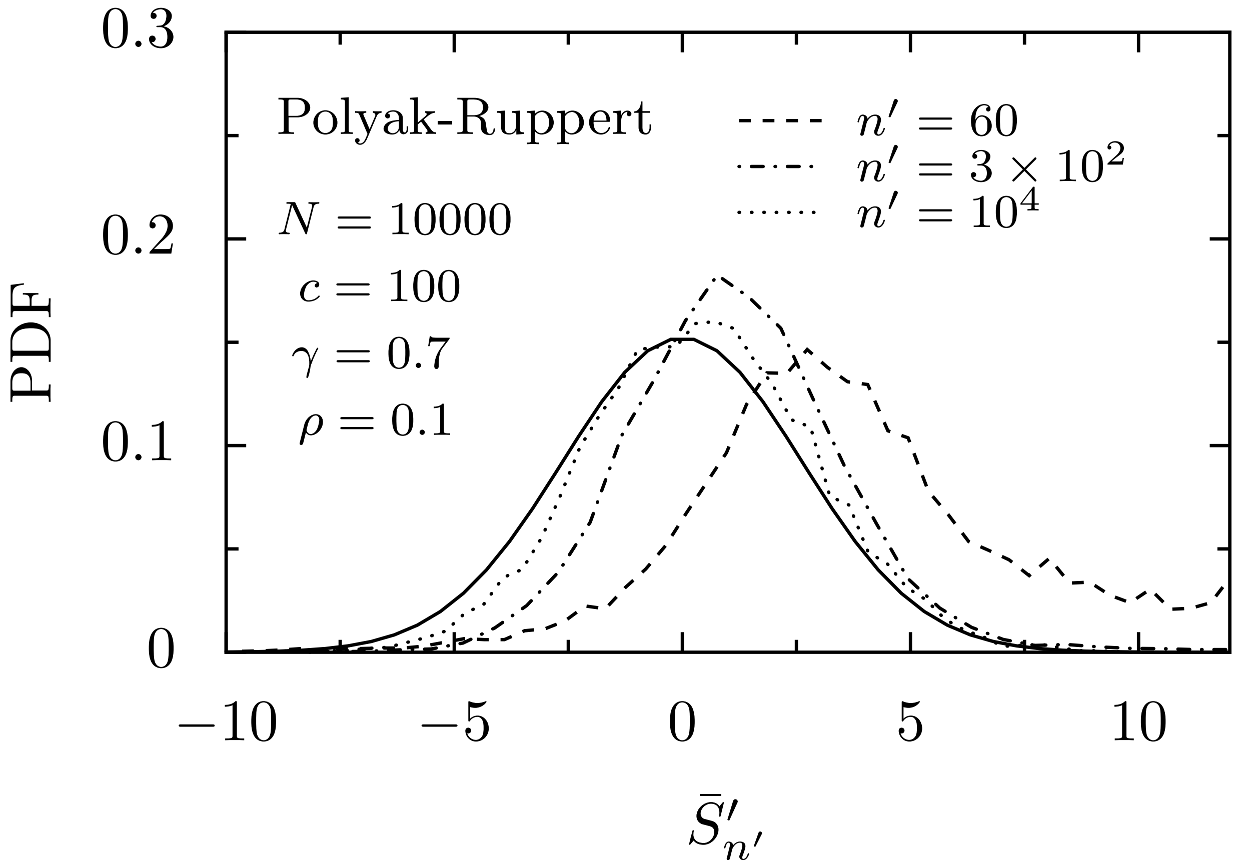
$$\bar{s}_n = \frac{1}{\rho \cdot n} \sum_{i=(1-\rho)n}^{n} s_i.$$

Then $\bar{s}_n \to s^*$ P*-almost surely. For every* $\varepsilon > 0$ *there exists another process* $\hat{s}$ *such that* $P(\bar{s}_n = \hat{s}_n \ \forall n) \ \geq \ 1 - \varepsilon$ *and*

$$\sqrt{\rho n} \cdot (\hat{s}_n - s^*) \to \mathcal{N}\left(0, \frac{\sigma^2(s^*)}{(f'(s^*))^2}\right).$$

- Optimal rate and asymptotic variance guaranteed
- Finite sample properties usually good

Stochastic approximation (3)



Averaging algorithmus: $\sqrt{\rho n}(\tilde{s}_n - s^*)$ is asymptotically normal (simulation of UBSR with polynomial loss function in the NCM with IS).

Conclusion

Conclusion

(i) Axiomatic theory of risk measures

- VaR is not a good risk measure
- Better risk measures have been designed, e.g. Utility-based Shortfall Risk

(ii) Implementation in credit portfolio models

- Importance sampling
- Stochastic approximation

Further research

- Comparison of stochastic average approximation with stochastic approximation
- Extension of the proposed techniques to a larger class of risk measures, e.g. optimized certainty equivalents
- Adjustments for liquidity risk
- Dynamic risk measurement procedures, and their numerical implementation

Thank you for your attention!