The Impact of Stochastic Volatility on Pricing, Hedging, and Hedge Efficiency of Variable Annuity Guarantees

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Introduction

- **Variable annuities are unit-linked deferred annuities**
  - In the US: Usually single premium contracts
  - Single premium is invested in fund(s)
  - In the 90s, insurance companies started to provide additional guarantees
    - Guaranteed Minimum Death Benefits (GMDB)
    - Different Guaranteed Minimum Survival Benefits
  - Fee for the guarantee: annually a certain percentage of the net asset value (NAV)
  - Guarantee provided by the insurance company
  - Risk management
    - “Reinsurance”
    - Internal hedging

Introduction

- **Guaranteed Minimum Death Benefits (GMDB)**
  - Death benefit = max {NAV ; guaranteed benefit base}
  - Typical forms of guaranteed benefit base
    - The premium paid by the policyholder
    - Maximum historical NAV of the fund at certain observation dates
      - e.g. once a year → annual ratchet guarantee
    - Annually increasing death benefit
      - Premium compounded at x% p.a.

- **Guaranteed Minimum Accumulation Benefits (GMAB)**
  - Survival benefit = max {NAV ; guaranteed benefit base}
  - Typical forms of guaranteed benefit base
    - Premium paid
    - Maximum historical NAV of the fund at certain observation dates
      - e.g. once a year → annual ratchet guarantee

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Introduction

- **Guaranteed Minimum Income Benefits (GMIB)**
  - Guaranteed (lifelong or temporary) annuity in case of annuitization
  - During some annuitization period, the policyholder may at any time
    - Annuitize the fund NAV at the current annuity conversion rate
    - Receive the fund NAV as a lump sum payment
    - Annuitize the guaranteed benefit base at an annuity conversion rate that has been guaranteed at t=0
  - Typical forms of the guaranteed benefit base
    - Maximum historical NAV of the fund
    - Premium compounded at x% p.a.

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Introduction

- **Guaranteed Minimum Withdrawal Benefits (GMWB)**
  - The policyholders can perform certain withdrawals from their fund, even if the value of the policy has dropped to zero
    - Usually: limit to withdrawals p.a. and sum of all withdrawals
- **Latest version and focus of this paper: "Guaranteed Lifetime Withdrawal Benefits"(GLWB)**
  - the policyholder is guaranteed lifelong minimum withdrawals (i.e. no limit on sum of all withdrawals)
    - however, the invested capital is not annuitized ➔ fund assets remain accessible
    - typically: guaranteed withdrawals increase in case of good fund performance
  - withdrawals are deducted from the policyholder's account value as long as it has not been depleted
    - afterwards, the insurer has to compensate for the guaranteed withdrawals (then essentially equivalent to a lifelong fixed annuity) until the insured's death
    - in return for this guarantee, the insurer receives guarantee fees deducted from the policyholder's fund assets
- **combination of policyholder behavior, longevity and market risk → difficult to hedge**

Research Questions

- **The focus of the paper lies on market risk and specifically on volatility risk in the context of GLWB options. Its key question is on model risk:**

  Compared to a model that assumes deterministic equity volatility, what impact does stochastic equity volatility have on pricing and hedging of GLWB options?

  - How do different (dynamic) hedging strategies perform under different data-generating models?
  - How are these effects influenced by the product design of the GLWB option?

Agenda

- **Product designs**
- Market models, Policyholder behavior and Pricing
- Hedging strategies
- Hedging results

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Product designs of the GLWB option

- All considered designs guarantee an annual minimum withdrawal amount for the lifetime of the insured.
  - surrender benefit = account value
    - in this case contract and guarantee end
  - death benefit = account value
    - in this case contract and guarantee end

- Depending on the product design, the guaranteed withdrawal amount can increase if the fund performs well.

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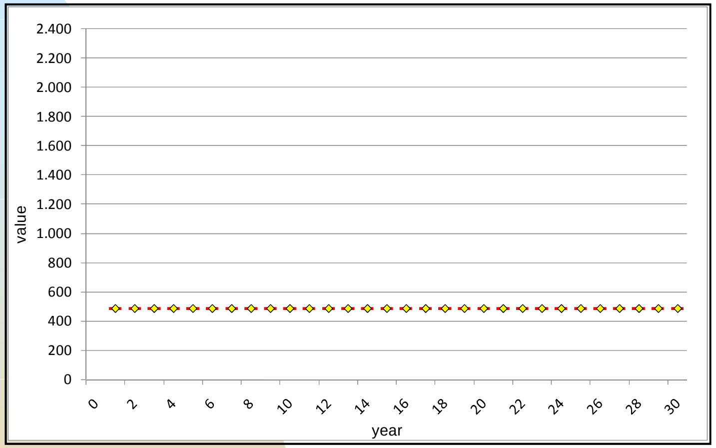
Product designs of the GLWB option

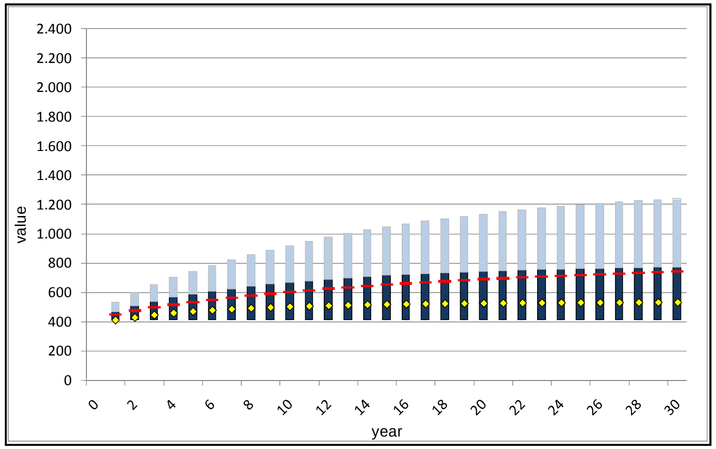
- **Four different ratchet mechanisms considered:**
  - No Ratchet
    - the guaranteed withdrawal amount remains constant
  - Lookback Ratchet
    - the guaranteed withdrawal amount is calculated as a percentage of the highest account value at all past policy anniversaries
  - Remaining Withdrawal Benefit Base (WBB) Ratchet
    - if the account value exceeds a certain reference value that decreases with each withdrawal, the difference is used to increase the guaranteed withdrawal amount for all following payments
  - Performance Bonus
    - if the account value exceeds a certain reference value, 50% of the difference is paid out to the policyholder immediately; future guarantees do not change

(Real-world) Distributions of the annual guaranteed withdrawal amount
(all four product designs priced with the same guarantee fee)

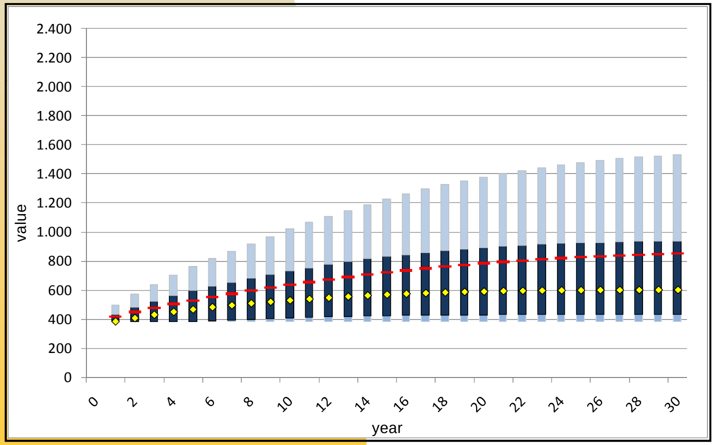
1 – No Ratchet



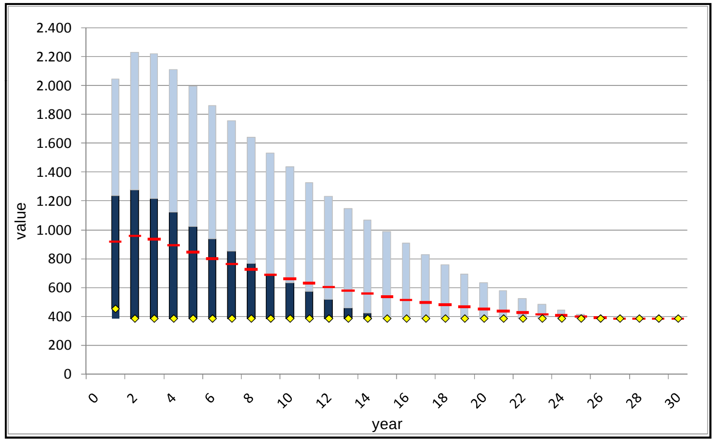
2 – Lookback Ratchet



3 – Remaining WBB Ratchet



4 – Performance Bonus



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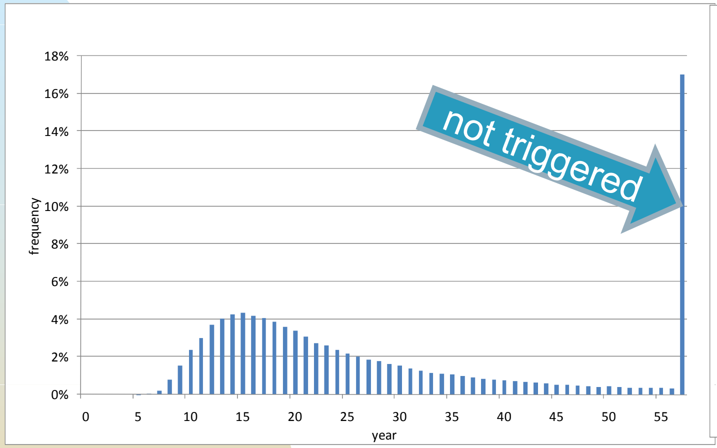
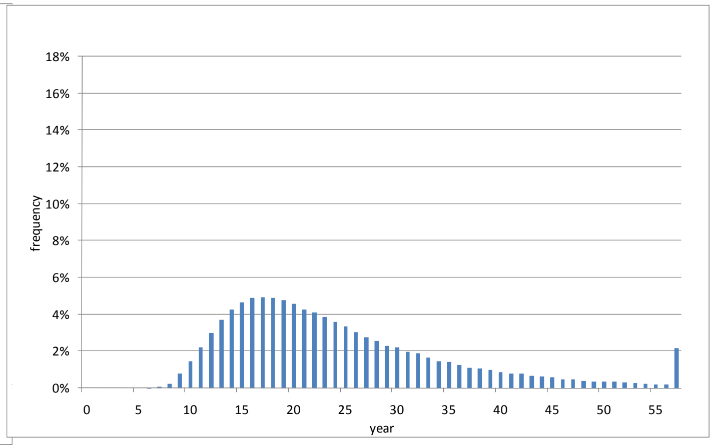
(Real-world) Distributions of the guarantee trigger time
(all four product designs priced with the same guarantee fee)

1 – No Ratchet



2 – Lookback Ratchet



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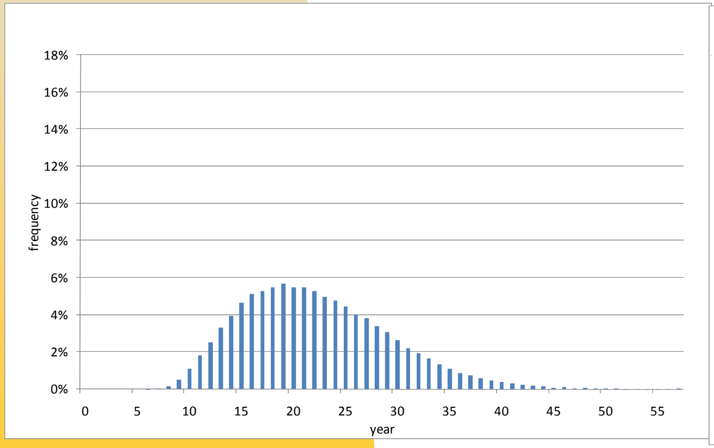
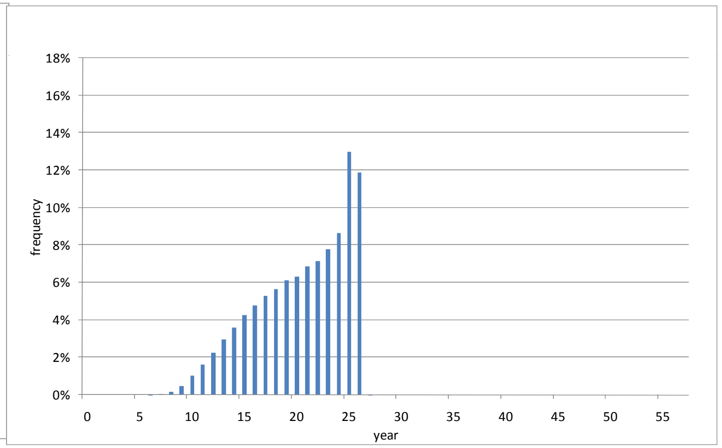


4 – Performance Bonus



Agenda

- Product designs
- **Market models, Policyholder behavior and Pricing**
- Hedging strategies
- Hedging results

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Market models used for pricing, hedging and simulation

- constant interest rates (4% in all numerical analyses)
- no spreads / no transaction costs
- The dynamics of the contract's underlying fund is given by :
  - **Black-Scholes (1973)**

$$dS(t) = \mu S(t)dt + \sigma_{BS} S(t)dW(t), \quad S(0) \geq 0$$

  - **Heston (1993)**

$$dS(t) = \mu S(t)dt + \sqrt{V(t)}S(t)dW_1\ (t), \quad S(0) \geq 0$$

$$dV(t) = \kappa\big(\theta - V(t)\big)dt + \sigma_v \sqrt{V(t)}dW_2\ (t), \quad V(0) \geq 0$$

  - where
    - μ - drift
      σ_{BS} - Black-Scholes volatility
    - $V(t)$ - local variance at time t
      κ - speed of mean reversion
      θ - long-term average variance
      σ_v - "vol of vol"
      $W_{1/2}$ - Wiener processes
      ρ - correlation between W1 and W2

Policyholder Behavior

- **only two possibilities considered:**
  - policyholder withdraws exactly the guaranteed amount
  - policyholder withdraws all of the remaining fund assets
    - ➔ full surrender

- **probabilistic policyholder behavior**
  - each year, a certain deterministic percentage of the policyholders perform full surrender
  - base case:

Year	Surrender rate p_t^S
1	6 %
2	5 %
3	4 %
4	3 %
5	2 %
≥ 6	1 %

Selected Pricing Results

- **“Fair” initial guaranteed withdrawal rate for a guarantee fee of 1.5% p.a. under the B-S-model**

Volatility	Surrender \ Ratchet mechanism	I (No Ratchet)	II (Lookback Ratchet)	III (Remaining WBB Ratchet)	IV (Performance Bonus)
σ_{BS}=15 %	*No surr*	5.26 %	4.80 %	4.43 %	4.37 %
	Surr 1 (base)	5.45 %	5.00 %	4.62 %	4.57 %
	*Surr 2 (base*2)*	5.66 %	5.22 %	4.83 %	4.79 %
σ_{BS}=20 %	*No surr*	4.98 %	4.32 %	4.01 %	4.00 %
	Surr 1 (base)	5.16 %	4.50 %	4.18 %	4.19 %
	*Surr 2 (base*2)*	5.35 %	4.71 %	4.38 %	4.40 %
σ_{BS}=22 %	*No surr*	4.87 %	4.13 %	3.85 %	3.85 %
	Surr 1 (base)	5.04 %	4.30 %	4.01 %	4.03 %
	*Surr 2 (base*2)*	5.23 %	4.50 %	4.20 %	4.24 %
σ_{BS}=25 %	*No surr*	4.70 %	3.85 %	3.61 %	3.62 %
	Surr 1 (base)	4.86 %	4.01 %	3.76 %	3.81 %
	*Surr 2 (base*2)*	5.04 %	4.20 %	3.94 %	4.01 %

As expected: Guaranteed withdrawal rate is
→ **decreasing in σ**
→**increasing in surrender rates**

Also: Guarantee value depends strongly on “richness” of ratchet mechanism

- **Further results:**
  - Pricing results under the Heston model: Very similar to B-S where B-S volatility coincides with average volatility in the Heston model
  - → For the pricing (as opposed to hedging), long-term volatility assumption is much more crucial than the question whether volatility is modeled stochastic or deterministic.

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Agenda

- Product designs
- Market models, Policyholder behavior and Pricing
- **Hedging strategies**
- Hedging results

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Hedging strategies

- **two hedging models: Black-Scholes and Heston**

- **three different types of dynamic hedging strategies considered:**
  - **no active hedging**
  - **hedging using the underlying only**
    - delta hedge in case of *Black-Scholes* **(D-BS)**
    - local risk minimizing hedge (LRM) in case of *Heston* **(D-H)**
  - **hedging using the underlying and a pre-specified (standard) option on the underlying**
    - aim: attain additional vega-neutrality of the portfolio
    - straightforward approach in the Heston case **(DV-H)**
    - two different approaches for vega hedging with Black-Scholes:
      - unmodified vega
        - first order derivative of option value with respect to B-S volatility
      - modified vega **(DV-BS (mod))**
        - sum of weighted vega of time t cash-flows $ModVega(\tau) = \sum_{t=\tau+1}^{T} \nu_t \frac{1}{\sqrt{t-\tau}}$

Agenda

- Product designs
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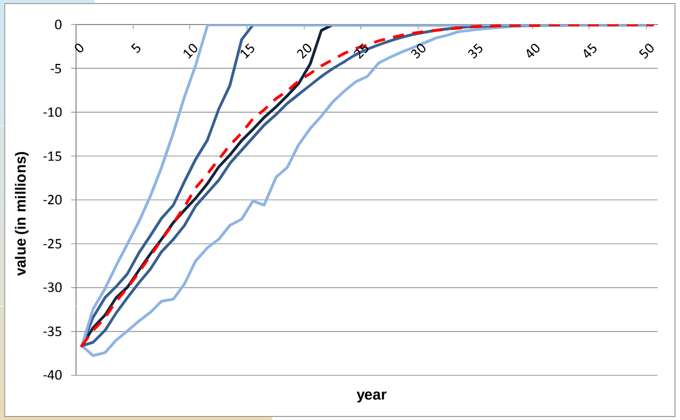
- **Hedging results**

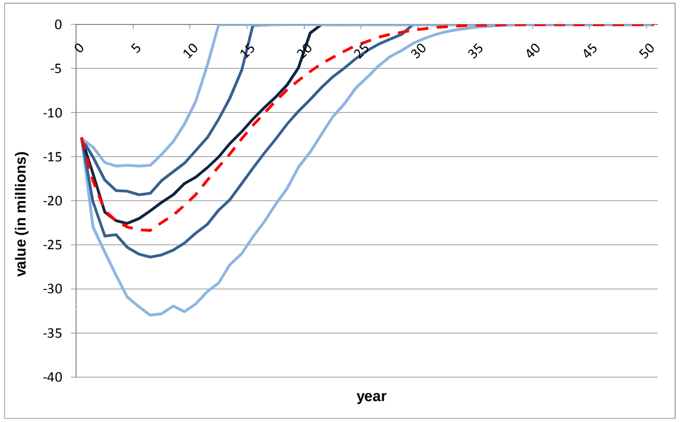
Hedging strategies – Significant difference between the products

- **Distribution of the Delta of the option for the different designs (under the B-S model)**

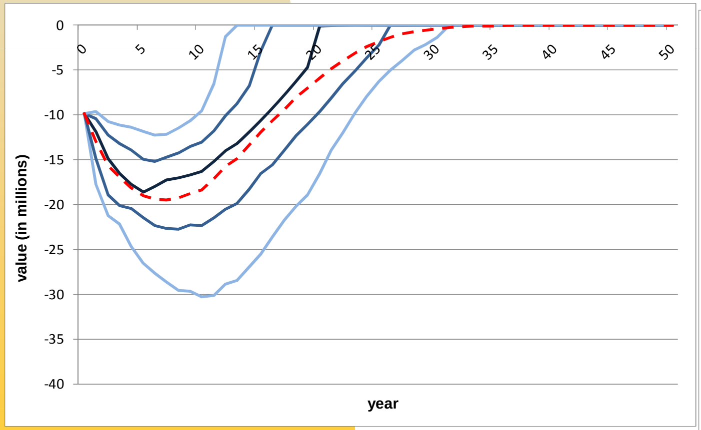
1 – No Ratchet



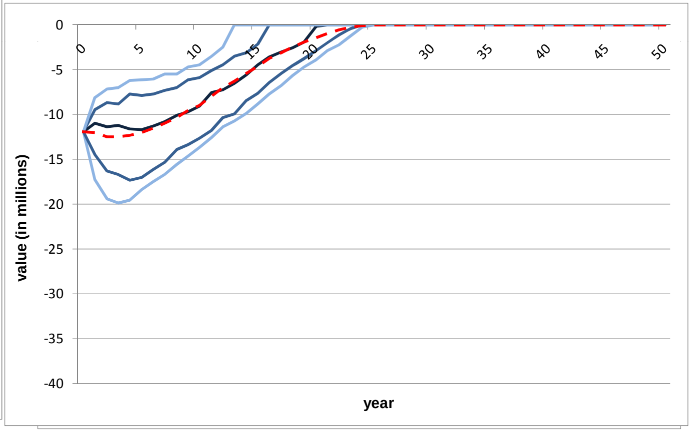
2 – Lookback Ratchet



3 – Remaining WBB Ratchet



4 – Performance Bonus



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Hedging – Some simulation results

Insurer's profitability and risk under different hedging strategies

		Data-Generating model							
		Black-Scholes				Heston			
		Product				Product			
		I	II	III	IV	I	II	III	IV
No hedge **(NH)**	Exp. Profit	10.43	7.77	6.67	3.88	10.36	7.97	6.82	4.13
	CTE loss (any time)	25.29	20.07	17.54	15.12	25.76	20.97	18.54	15.97
	CTE final loss	23.41	18.27	15.90	13.35	22.93	18.55	16.25	13.51
Delta hedge Black-Scholes **(D-BS)**	Exp. Profit	0.48	0.27	0.21	0.17	0.57	0.29	0.17	0.13
	CTE loss (any time)	1.71	3.25	3.12	2.02	2.77	4.76	4.51	3.35
	CTE final loss	1.44	2.74	2.71	1.78	2.44	4.14	3.99	3.02
LRM Heston **(D-H)**	Exp. Profit	n/a				0.52	0.42	0.34	0.21
	CTE loss (any time)					2.63	4.59	4.44	3.36
	CTE final loss					2.28	4.03	3.98	2.95
Delta-Vega hedge Black-Scholes (mod) **(DV-BS)**	Exp. Profit	n/a				0.82	0.81	0.75	0.47
	CTE loss (any time)					1.75	2.41	3.01	1.88
	CTE final loss					1.35	1.80	2.40	1.53
Delta-Vega hedge Heston **(DV-H)**	Exp. Profit	n/a				0.49	0.41	0.33	0.19
	CTE loss (any time)					1.40	1.99	1.95	1.49
	CTE final loss					1.15	1.58	1.60	1.21

Hedging – Some simulation results

- **Insurer's profitability and risk under different hedging strategies**

		Data-Generating model: Black-Scholes, Product I	Black-Scholes II	Black-Scholes III	Black-Scholes IV	Heston, Product I	Heston II	Heston III	Heston IV
No hedge **(NH)**	Exp. Profit	10.43	7.77	[illegible]	3.88	[illegible]	7.97	6.82	4.13
	CTE loss (any time)	25.29	20.07	[illegible]	15.12	25.76	[illegible]	18.54	15.97
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Black-Scholes delta hedge:

data-generating model:
B-S➔ Heston:
Risk +47% to +70%

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Hedging – Some simulation results

Insurer's profitability and risk under different hedging strategies

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		Black-Scholes				Heston			
		Product				Product			
		I	II	III	IV	I	II	III	IV
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Heston data-generating model:

D-BS → D-H:
Risk 0% to -7%

D-BS → DV-BS:
Risk -40% to -57%

D-H → DV-H:
Risk -50% to -61%

Hedging – Some simulation results

- **Finally: Some words on vega hedging in a BS-framework:**
  - As seen on the previous slide:
    - Using modified vega in a B-S framework can significantly reduce the risk stemming from stochastic volatility, even if a model with deterministic volatility is used for hedging
  - However, if a somewhat more intuitive approach (unmodified vega) is used, the risk may increase dramatically:
    - B-S delta ➔ B-S delta-vega (unmod. vega): +166% to +282%

Thank you for your attention!

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