Time-Consistent and Market-Consistent Actuarial Valuations

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30 April 2010
DGVFM Scientific Day – Bremen

Motivation

- Standard actuarial premium principles usually consider "static" premium calculation:
  - What is price today of insurance contract with payoff at time T?
- Actuarial premium principles typically "ignore" financial markets
- Financial pricing considers "dynamic" pricing problem:
  - How does price evolve over time until time T?
- Financial pricing typically "ignores" unhedgeable risks
- Examples:
  - Pricing very long-dated cash flows $T \sim 30 - 100$ years
  - Pricing long-dated options $T > 5$ years
  - Pricing pension & insurance liabilities
  - Pricing employee stock-options

Ambitions

- In this paper I want to combine
  1. Time-Consistent pricing operators, see [Jobert and Rogers, 2008]
  2. Market-Consistent pricing operators, see [Malamud et al., 2008]
- Both references concentrate on discrete-time algorithms
- I will be interested in continuous-time limits of these discrete algorithms for different actuarial premium principles:
  1. Variance Principle
  2. Mean Value Principle
  3. Standard-Deviation Principle
  4. Cost-of-Capital Principle

Content of This Talk

1. Pure Insurance Risk
   - Diffusion Model for Insurance Risk
   - Variance Principle ($\rightarrow$ exponential indiff. pricing)
   - Standard-Dev. Principle ($\rightarrow \mathbb{E}[]$ under new measure)
   - Cost-of-Capital Principle ($\rightarrow$ St.Dev price)
   - Davis Price, see [Davis, 1997]
     - St.Dev is "small perturbation" of Variance price
2. Financial & Insurance Risk
   - Diffusion Model for Financial Risk
   - Market-Consistent Pricing
   - Variance Principle
   - Numerical Illustration
3. Conclusions

Pure Insurance Model

- Consider unhedgeable insurance process y:

$$dy = a(t, y)\, dt + b(t, y)\, dW$$

- To keep math simple, concentrate on diffusion setting
- Discretisation scheme as binomial tree:

$$y(t + \Delta t) = y(t) + a\Delta t + \begin{cases} +b\sqrt{\Delta t} & \text{with prob. } \frac{1}{2} \\ -b\sqrt{\Delta t} & \text{with prob. } \frac{1}{2} \end{cases}$$

Time Consistency

- Time Consistent price $\pi(t, y)$ satisfies property

$$\pi[f(y(T))|t, y] = \pi[\pi[f(y(T))|s, y(s)]|t, y] \quad \forall t < s < T$$

- Price of today of holding claim until T is the same as buying claim half-way at time s for price $\pi(s, y(s))$
- "Semi-group property"
- Similar idea as "tower property" of conditional expectation

Variance Principle

- Actuarial Variance Principle Π^{v}:

$$\Pi_t^{\mathsf{v}}[f(y(T))] = \mathbb{E}_t[f(y(T))] + \tfrac{1}{2}\alpha \mathbb{V}\mathrm{ar}_t[f(y(T))]$$

- α is Absolute Risk Aversion
- Apply Π^{v} to one binomial time-step to obtain price π^{v}:

$$\pi^{\mathsf{v}}\big(t, y(t)\big) = \mathbb{E}_t[\pi^{\mathsf{v}}\big(t + \Delta t, y(t + \Delta t)\big)] + \tfrac{1}{2}\alpha \mathbb{V}\mathrm{ar}_t[\pi^{\mathsf{v}}\big(t + \Delta t, y(t + \Delta t)\big)]$$

- Note: we omit discounting for now

Pricing PDE

- Assume $\pi^{\mathsf{v}}(t, y)$ admits Taylor approximation in y
- Evaluate Var.Princ. for binomial step & take limit for $\Delta t \to 0$
  - Same as derivation of Feynman-Kaç, but for $\mathbb{E}[] + \frac{1}{2}\alpha\mathbb{V}\mathrm{ar}[]$
- This leads to pde for π^{v}:

$$\pi^{\mathsf{v}}_t + a\pi^{\mathsf{v}}_y + \tfrac{1}{2}b^2\pi^{\mathsf{v}}_{yy} + \tfrac{1}{2}\alpha(b\pi^{\mathsf{v}}_y)^2 = 0$$

- Note, non-linear term = "local unhedgeable variance" $b^2(\pi^{\mathsf{v}}_y)^2$
- Find general solution to this non-linear pde via log-transform:

$$\pi^{\mathsf{v}}(t, y) = \frac{1}{\alpha} \ln \mathbb{E}_t \left[e^{\alpha f(y(T))} \middle| y(t) = y \right].$$

- Exponential indifference price, see [Henderson, 2002] or [Musiela and Zariphopoulou, 2004]

Include Discounting

- We should include discounting into our pricing
- Absolute Risk Aversion α is not "unit-free", but has unit $1/€$
  - This conveniently compensates the unit $(€)^2$ of $\mathbb{V}\text{ar}[]\ldots$
- Therefore, "α-today" is different than "α-tomorrow"
- Relative Risk Aversion γ is unit-free
- Express ARA relative to "benchmark wealth" $X_0 e^{rT}$
- Explicit notation: $\alpha \to \gamma / X_0 e^{rT}$ leads to pde:

$$\pi_t^{\mathsf{v}} + a\pi_y^{\mathsf{v}} + \tfrac{1}{2} b^2 \pi_{yy}^{\mathsf{v}} + \tfrac{1}{2} \frac{\gamma}{X_0 e^{rt}} (b\pi_y^{\mathsf{v}})^2 - r\pi^{\mathsf{v}} = 0$$

$$\pi^{\mathsf{v}}(t, y) = \frac{X_0 e^{rt}}{\gamma} \ln \mathbb{E}\left[e^{\frac{\gamma}{X_0 e^{rT}} f(y(T))} \middle| y(t) = y \right]$$

- Note: express all prices in *discounted* terms

Backward Stochastic Differential Equations

- Pricing PDE:

$$\pi^{\mathsf{v}}_t + a\pi^{\mathsf{v}}_y + \tfrac{1}{2}b^2\pi^{\mathsf{v}}_{yy} + \tfrac{1}{2}\frac{\gamma}{X_0 e^{rt}}(b\pi^{\mathsf{v}}_y)^2 - r\pi^{\mathsf{v}} = 0$$

- This non-linear PDE, represents the solution to a so-called BSDE for the triplet of processes (y_t, Y_t, Z_t)

$$\begin{cases} dy_t = a(t, y_t)\,dt + b(t, y_t)\,dW_t \\ dY_t = -g(t, y_t, Y_t, Z_t)\,dt + Z_t\,dW_t \\ Y_T = f\big(y(T)\big), \end{cases}$$

- with "generator" $g(t, y, Y, Z) = \frac{1}{2}\frac{\gamma}{X_0 e^{rt}}Z^2 - rY$.
- Recent literature studies uniqueness & existence of solutions to BSDE's, see [El Karoui et al., 1997]
- Via BSDE's we can study time-consistent pricing operators in a much more general stochastic setting. But we will not pursue this here.

Mean Value Principle

- Generalise to *Mean Value Principle*

$$\Pi_t^{\mathrm{m}}[f(y(T))] = v^{-1}\left(\mathbb{E}_t[v(f(y(T)))]\right)$$

- for any function $v()$ which is a convex and increasing
    - Exponential pricing is special case with $v(x) = e^{\alpha x}$
- Do Taylor-expansion & limit $\Delta t \to 0$:

$$\pi_t^{\mathrm{mf}} + a\pi_y^{\mathrm{mf}} + \tfrac{1}{2}b^2\pi_{yy}^{\mathrm{mf}} + \tfrac{1}{2}\frac{v''(\pi^{\mathrm{mf}})}{v'(\pi^{\mathrm{mf}})}(b\pi_y^{\mathrm{mf}})^2 = 0$$

- Note: $\pi^{\mathrm{mf}}(t,y) := \pi^{\mathrm{m}}(t,y)/e^{rt}$ is price expressed in discounted terms
- Interpretation as generalised Variance Principle with "local risk aversion" term: $v''()/v'()$

Standard-Deviation Principle

- Actuarial Standard-Deviation Principle:

$$\Pi_t^s[f(y(T))] = \mathbb{E}_t[f(y(T))] + \beta\sqrt{\mathbb{V}\mathrm{ar}_t[f(y(T))]}.$$

- Pay attention to "time-scales":
  - Expectation scales with Δt
  - St.Dev. scales with $\sqrt{\Delta t}$
- Thus, we should take $\beta\sqrt{\Delta t}$ to get well-defined limit
  - Note: β has unit $1/\sqrt{\text{time}}$

Pricing PDE

- Do Taylor-expansion & limit $\Delta t \to 0$:

$$\pi^{\mathsf{s}}_t + a\pi^{\mathsf{s}}_y + \tfrac{1}{2}b^2\pi^{\mathsf{s}}_{yy} + \beta\sqrt{(b\pi^{\mathsf{s}}_y)^2} - r\pi^{\mathsf{s}} = 0$$

- Again, non-linear pde. But if π^{s} is monotone in y then

$$\pi^{\mathsf{s}}_t + (a \pm \beta b)\pi^{\mathsf{s}}_y + \tfrac{1}{2}b^2\pi^{\mathsf{s}}_{yy} - r\pi^{\mathsf{s}} = 0$$

$$\pi^{\mathsf{s}}(t, y) = \mathbb{E}^{\$}_t\left[f(y(T))|y(t) = y\right]$$

- "Upwind" drift-adjustment into direction of risk

Cost-of-Capital Principle

- Cost-of-Capital principle, popular by practitioners
  - Used in QIS4-study conducted by CEIOPS
- Idea: hold buffer-capital against unhedgeable risks. Borrow from shareholders by giving "excess return" δ
- Define buffer via Value-at-Risk measure:

$$\Pi_t^c[f(y(T))] = \mathbb{E}_t[f(y(T))] + \delta \mathrm{VaR}_{q,t}\Big[f(y(T)) - \mathbb{E}_t[f(y(T))]\Big].$$

Scaling & PDE

- Again, pay attention to "time-scaling":
  - First, scale VaR back to *per annum* basis with $1/\sqrt{\Delta t}$
  - Then, δ is like interest rate, so multiply with Δt
  - Net scaling: $\delta\Delta t/\sqrt{\Delta t} = \delta\sqrt{\Delta t}$.
- Limit: for small Δt the VaR behaves as $\Phi^{-1}(q)\times$St.Dev. Hence, limiting pde is same as π^{s} but with $\beta = \Phi^{-1}(q)\delta$.

- Conclusion: In the limit for $\Delta t \to 0$, CoC pricing is the same as st.dev. pricing

Davis Price

- The variance price π^{v} is "hard" to calculate, the st.dev. price π^{s} is "easy" to calculate
- Can we make a connection between these two concepts?
- "Yes, we can!" using small perturbation expansion
- Consider existing insurance portfolio with price $\pi^{\mathsf{v}}(t, y)$, now add "small" position with price $\varepsilon\pi^{\mathsf{D}}(t, y)$. Subst. into pde:

$$(\pi_t^{\mathsf{v}} + \varepsilon\pi_t^{\mathsf{D}}) + a(\pi_y^{\mathsf{v}} + \varepsilon\pi_y^{\mathsf{D}}) + \tfrac{1}{2}b^2(\pi_{yy}^{\mathsf{v}} + \varepsilon\pi_{yy}^{\mathsf{D}}) + \\ \tfrac{1}{2}\frac{\gamma}{X_0 e^{rt}}b^2\left((\pi_y^{\mathsf{v}})^2 + 2\varepsilon\pi_y^{\mathsf{v}}\pi_y^{\mathsf{D}} + \varepsilon^2(\pi_y^{\mathsf{D}})^2\right) - r(\pi^{\mathsf{v}} + \varepsilon\pi^{\mathsf{D}}) = 0$$

- $\pi^{\mathsf{v}}()$ solves the pde, cancel π^{v}-terms

Pricing PDE

- Simplify pde, and divide by ε:

$$\pi_t^{\mathrm{D}} + a\pi_y^{\mathrm{D}} + \tfrac{1}{2} b^2 \pi_{yy}^{\mathrm{D}} + \tfrac{1}{2} \frac{\gamma}{X_0 e^{rt}} b^2 \left(2\pi_y^{\mathrm{v}} \pi_y^{\mathrm{D}} + \varepsilon (\pi_y^{\mathrm{D}})^2 \right) - r\pi^{\mathrm{D}} = 0$$

- Approximation: ignore "small" ε-term

$$\pi_t^{\mathrm{D}} + \left(a + \frac{\gamma}{X_0 e^{rt}} b^2 \pi_y^{\mathrm{v}} \right) \pi_y^{\mathrm{D}} + \tfrac{1}{2} b^2 \pi_{yy}^{\mathrm{D}} + r\pi^{\mathrm{D}} = 0$$

$$\pi^{\mathrm{D}}(t, y) = \mathbb{E}_t^{\mathbb{D}} \left[f(y(T)) | y(t) = y \right]$$

- Davis price π^{D} is defined only "relative" to existing price π^{v} of insurance portfolio
- Note, drift-adjustment of st.dev. price scales with b

- Investigate environment with financial risk that can be traded (and hedged!) in financial market *and* non-traded insurance risk
- Model financial risk as [Black and Scholes, 1973] economy. Model return process $x_t = \ln S_t$ under real-world measure $\mathbb{P}$:

$$dx = \big(\mu(t,x) - \tfrac{1}{2}\sigma^2(t,x)\big)\,dt + \sigma(t,x)\,dW_f$$

- Binomial time-step:

$$x(t+\Delta t) = x(t) + (\mu - \tfrac{1}{2}\sigma^2)\Delta t + \begin{cases} +\sigma\sqrt{\Delta t} & \text{with } \mathbb{P}\text{-prob. } \tfrac{1}{2} \\ -\sigma\sqrt{\Delta t} & \text{with } \mathbb{P}\text{-prob. } \tfrac{1}{2} \end{cases}$$

No-arbitrage pricing

- BS-economy is arbitrage-free and complete $\Leftrightarrow$ unique martingale measure $\mathbb{Q}$.
- No-arbitrage pricing operator for financial derivative $F(x(T))$:

$$\pi^{\mathbb{Q}}(t,x) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[F(x(T))]$$

- Binomial step for x under measure $\mathbb{Q}$:

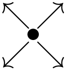
$$x(t+\Delta t) = x(t) + (\mu - \tfrac{1}{2}\sigma^2)\Delta t + \begin{cases} +\sigma\sqrt{\Delta t} & \text{with } \mathbb{Q}\text{-prob. } \frac{1}{2}\left(1 - \frac{\mu-r}{\sigma}\sqrt{\Delta t}\right) \\ -\sigma\sqrt{\Delta t} & \text{with } \mathbb{Q}\text{-prob. } \frac{1}{2}\left(1 + \frac{\mu-r}{\sigma}\sqrt{\Delta t}\right) \end{cases}$$

- Quantity $(\mu - r)/\sigma$ is Radon-Nikodym exponent of $d\mathbb{Q}/d\mathbb{P}$
- Quantity $(\mu - r)/\sigma$ is also known as *market-price of financial risk*.

Quadrinomial Tree

- Joint discretisation for processes x and y using "quadrinomial" tree with correlation ρ under measure $\mathbb{P}$:

State:	$y+\Delta y$	$y-\Delta y$
$x+\Delta x$	$\left(\frac{1+\rho}{4}\right)$	$\left(\frac{1-\rho}{4}\right)$
$x-\Delta x$	$\left(\frac{1-\rho}{4}\right)$	$\left(\frac{1+\rho}{4}\right)$



- Positive correlation increases probability of joint "++" or "−−" co-movement

Market-Consistent Pricing

- We are looking for *market-consistent* pricing operators, see e.g. [Malamud et al., 2008]

Definition

A pricing operator $\pi()$ *is market-consistent if for any financial derivative* $F(x(T))$ *and any other claim* $G(t, x, y)$ *we have*

$$\pi_{F+G}(t, x, y) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[F(x(T))] + \pi_G(t, x, y).$$

- Observation: generalised notion of "translation invariance" for all financial risks

MC Variance Pricing

- Intuition: construct MC pricing in two steps using conditional expectations, see also: [Carmona, 2008], Chap. 1
- First: condition on financial risk & use actuarial pricing for "pure insurance" risk

$$\pi^{\mathsf{v}}(t+\Delta t|x\pm) := \mathbb{E}[\pi^{\mathsf{v}}(t+\Delta t)|x\pm] + \frac{1}{2}\frac{\gamma}{X_0 e^{r(t+\Delta t)}}\mathbb{V}\mathrm{ar}[\pi^{\mathsf{v}}(t+\Delta t)|x\pm]$$

$$\begin{aligned}
\mathbb{E}[\pi^{\mathsf{v}}(t+\Delta t)|x+] &= \left(\tfrac{1+\rho}{2}\right)\pi^{\mathsf{v}}_{++} + \left(\tfrac{1-\rho}{2}\right)\pi^{\mathsf{v}}_{+-} \\
\mathbb{V}\mathrm{ar}[\pi^{\mathsf{v}}(t+\Delta t)|x+] &= \left(\tfrac{1-\rho^2}{4}\right)\left(\pi^{\mathsf{v}}_{++} - \pi^{\mathsf{v}}_{+-}\right)^2 \\
\mathbb{E}[\pi^{\mathsf{v}}(t+\Delta t)|x-] &= \left(\tfrac{1-\rho}{2}\right)\pi^{\mathsf{v}}_{-+} + \left(\tfrac{1+\rho}{2}\right)\pi^{\mathsf{v}}_{--} \\
\mathbb{V}\mathrm{ar}[\pi^{\mathsf{v}}(t+\Delta t)|x-] &= \left(\tfrac{1-\rho^2}{4}\right)\left(\pi^{\mathsf{v}}_{-+} - \pi^{\mathsf{v}}_{--}\right)^2 .
\end{aligned}$$

- For $\rho = 1$ or $\rho = -1$ no unhedgeable risk left $\Rightarrow \mathbb{V}\mathrm{ar} = 0$

Pricing PDE

- Second: use no-arbitrage pricing for "artificial" financial derivative

$$\begin{aligned}\pi^{\mathsf{v}}(t,x,y) &= e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}[\pi^{\mathsf{v}}(t+\Delta t|x\pm)] \\ &= e^{-r\Delta t}\left(\tfrac{1}{2}\left(1-\tfrac{\mu-r}{\sigma}\sqrt{\Delta t}\right)\pi^{\mathsf{v}}(t+\Delta t|x+)+\right. \\ &\qquad \left.\tfrac{1}{2}\left(1+\tfrac{\mu-r}{\sigma}\sqrt{\Delta t}\right)\pi^{\mathsf{v}}(t+\Delta t|x-)\right)\end{aligned}$$

- Do Taylor-expansion & limit $\Delta t \to 0$:

$$\begin{aligned}&\pi^{\mathsf{v}}_t + (r-\tfrac{1}{2}\sigma^2)\pi^{\mathsf{v}}_x + \left(a-\rho b\tfrac{\mu-r}{\sigma}\right)\pi^{\mathsf{v}}_y + \\ &\quad \tfrac{1}{2}\sigma^2\pi^{\mathsf{v}}_{xx} + \rho\sigma b\pi^{\mathsf{v}}_{xy} + \tfrac{1}{2}b^2\pi^{\mathsf{v}}_{yy} + \tfrac{1}{2}\frac{\gamma}{X_0 e^{rt}}(1-\rho^2)(b\pi^{\mathsf{v}}_y)^2 - r\pi^{\mathsf{v}} = 0\end{aligned}$$

- Impact on x: "$\mathbb{Q}$-drift" $(r-\frac{1}{2}\sigma^2)$
- Impact on y: adjusted drift $\left(a-\rho b\frac{\mu-r}{\sigma}\right)$
- Non-linear term for "locally unhedgeable variance" $(1-\rho^2)(b\pi^{\mathsf{v}}_y)^2$

Pure Insurance Payoff

- Unfortunately, we cannot solve the non-linear pde in general
- Special case: consider pure insurance payoff (and constant MPR $\frac{\mu-r}{\sigma}$), then no (explicit) dependence on x

$$\pi^{\mathrm{v}}_t + \left(a - \rho b \tfrac{\mu-r}{\sigma}\right) \pi^{\mathrm{v}}_y + \tfrac{1}{2} b^2 \pi^{\mathrm{v}}_{yy} + \tfrac{1}{2} \frac{\gamma}{X_0 e^{rt}} (1-\rho^2)(b\pi^{\mathrm{v}}_y)^2 - r\pi^{\mathrm{v}} = 0$$

- Note that for $\rho \neq 0$ the pde is different from the “pure insurance” pde
- Via correlation with financial market we can still hedge part of the insurance risk
- Note: incomplete market $\Rightarrow$ *no* martingale representation, therefore delta-hedge is *not* π^{v}_x
  - In fact: hold $\rho b \pi^{\mathrm{v}}_y/\sigma$ in x as hedge
  - Economic explanation for drift-adjustment in y, a kind of “quanto-adjustment”

Pure Insurance Payoff (2)

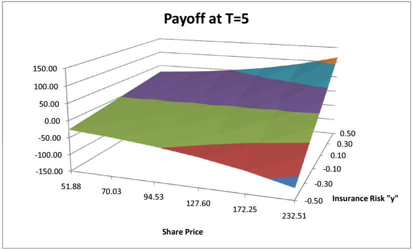
- General solution via log-transform:

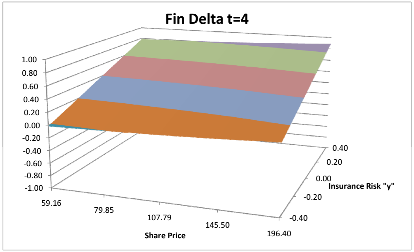
$$\pi^{v}(t,y) = \frac{X_0 e^{rt}}{\gamma(1-\rho^2)} \ln \mathbb{E}^{\tilde{\mathbb{P}}} \left[e^{\frac{\gamma(1-\rho^2)}{X_0 e^{rT}} f(y(T))} \middle| y(t) = y \right]$$

- Measure $\tilde{\mathbb{P}}$ induces drift-adjusted process for y
- See also, [Henderson, 2002] and [Musiela and Zariphopoulou, 2004] who derived this solution in the context of exponential indifference pricing
- We can generalise to Mean Value Principle for any convex function $v()$

Numerical Illustration

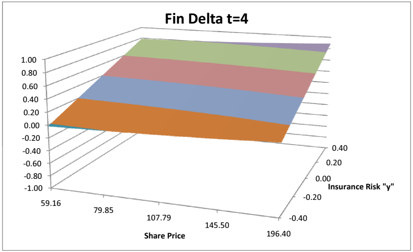
- Consider "unit-linked" insurance contract with payoff: $y(T)S(T) = y(T)e^{x(T)}$
- Numerical calculation in quadrinomial tree with 5 time-steps of 1 year
- "Naive" hedge is to hold $y(t)$ units of share $S(t)$
- In fact: hold $\pi_x^v + \rho b \pi_y^v / \sigma$ as hedge
- MC Variance hedge also builds additional reserve as "buffer" against unhedgeable risk

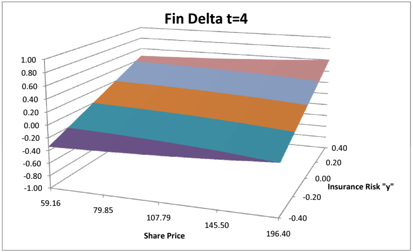




Numerical Illustration (2)

- Investigate impact of correlation ρ
- Compare $\rho = 0.50$ (left) and $\rho = 0$ (right)





- Positive correlation leads to higher delta, as this also hedges part of insurance risk: hold $\pi_x^{\mathsf{v}} + \rho b \pi_y^{\mathsf{v}} / \sigma$ as hedge
- Price for $\rho = 0.00$ at $t = 0$ is €26.75
- Price for $\rho = 0.50$ at $t = 0$ is €18.79, due to less unhedgeable risk
- Price for $\rho = 0.99$ at $t = 0$ is $-$€1.98, due to drift-adjustment

MC Standard-Deviation Pricing

- Again, do two-step construction
- First: condition on financial risk & use actuarial pricing for "pure insurance" risk

$$\pi^{\mathsf{s}}(t+\Delta t|x\pm) := \mathbb{E}[\pi^{\mathsf{s}}(t+\Delta t)|x\pm] + \delta\sqrt{\Delta t}\sqrt{\mathbb{V}\mathrm{ar}[\pi^{\mathsf{s}}(t+\Delta t)|x\pm]}$$

- Second: do no-arbitrage valuation under $\mathbb{Q}$
- This leads to linear pricing pde (if $\pi^{\mathsf{s}}(t,x,y)$ monotone in y):

$$\pi^{\mathsf{s}}_t + (r - \tfrac{1}{2}\sigma^2)\pi^{\mathsf{s}}_x + \left(a - \rho b\tfrac{\mu-r}{\sigma} \pm \delta\sqrt{1-\rho^2}\, b\right)\pi^{\mathsf{s}}_y + \\ \tfrac{1}{2}\sigma^2\pi^{\mathsf{s}}_{xx} + \rho\sigma b\pi^{\mathsf{s}}_{xy} + \tfrac{1}{2}b^2\pi^{\mathsf{s}}_{yy} - r\pi^{\mathsf{s}} = 0$$

- Drift adjustment for y is now combination of "hedge cost" plus "upwind" risk-adjustment $\pm\delta\sqrt{1-\rho^2}\, b$

MC Davis Price

- We can again consider Davis price, by "small perturbation" expansion
- This leads to pricing pde:

$$\pi^{\mathsf{D}}_t + (r - \tfrac{1}{2}\sigma^2)\pi^{\mathsf{D}}_x + \left(a - \rho b \tfrac{\mu - r}{\sigma} + \tfrac{\gamma}{X_0 e^{rt}}(1-\rho^2) b^2 \pi^{\mathsf{v}}_y\right)\pi^{\mathsf{D}}_y + \frac{1}{2}\sigma^2\pi^{\mathsf{D}}_{xx} + \rho\sigma b \pi^{\mathsf{D}}_{xy} + \frac{1}{2} b^2 \pi^{\mathsf{D}}_{yy} - r\pi^{\mathsf{D}} = 0$$

- Drift adjustment for y is now combination of "hedge cost" plus risk-adjustment $\frac{\gamma}{X_0 e^{rt}}(1-\rho^2) b^2 \pi^{\mathsf{v}}_y$
- Davis price defined relative to existing price $\pi^{\mathsf{v}}(t, x, y)$
- Note, st.dev. pricing depends on $\sqrt{(1-\rho^2)}\, b$

Multi-dimensional MC Variance Price

- Vectors x of asset returns (n-vector), y of insurance risks (m-vector)

$$dx = \mu\, dt + \Sigma^{\frac{1}{2}} \cdot dW_f$$

$$dy = a\, dt + B^{\frac{1}{2}} \cdot dW$$

- Partitioned covariance matrix C $(n+m) \times (n+m)$
  - P is $n \times m$ matrix of financial & insurance covariances

$$C = \begin{pmatrix} \Sigma & P \\ P' & B \end{pmatrix}$$

- The market-price of financial risks is an n-vector $\Sigma^{-1}(\mu - r)$
- Multi-dim pricing pde for π^{v}:

$$\pi^{\mathsf{v}}_t + r'\pi^{\mathsf{v}}_x + \Big(a - P'\Sigma^{-1}(\mu - r)\Big)'\pi^{\mathsf{v}}_y + \tfrac{1}{2}\left(C_{ij}\pi^{\mathsf{v}}_{ij}\right) + \tfrac{1}{2}\frac{\gamma}{X_0 e^{rt}}\Big(\pi^{\mathsf{v}\prime}_y(B - P'\Sigma^{-1}P)\pi^{\mathsf{v}}_y\Big) - r\pi^{\mathsf{v}} = 0$$

- Note: $(B - P'\Sigma^{-1}P)$ is conditional covariance matrix of $y|x$

Multi-dimensional MC StDev Price

- Multi-dim pricing pde for π^{s}:

$$\pi^{\mathsf{s}}_t + r'\pi^{\mathsf{s}}_x + \left(a - P'\Sigma^{-1}(\mu - r)\right)'\pi^{\mathsf{s}}_y + \tfrac{1}{2}\left(C_{ij}\pi^{\mathsf{s}}_{ij}\right) + \tfrac{1}{2}\delta\sqrt{\pi^{\mathsf{s}}_y{}'(B - P'\Sigma^{-1}P)\pi^{\mathsf{s}}_y} - r\pi^{\mathsf{s}} = 0$$

- Note: unlike 1-dim case, does not simplify to linear pde
- Simplification only possible if $(B - P'\Sigma^{-1}P)$ has rank 1 *and* all π^{s}_i have same sign

Conclusions

1. Pure Insurance Risk
   - Variance Principle ($\rightarrow$ exponential indiff. pricing)
   - Standard-Dev. Principle ($\rightarrow \mathbb{E}[]$ under new measure)
   - Cost-of-Capital Principle ($\rightarrow$ St.Dev price)
   - Davis Price: St.Dev. price is "small perturbation" of Variance price
2. Financial & Insurance Risk
   - Market-Consistent Pricing: via two-step conditional expectations
   - MC Variance Principle
   - Numerical Illustration for "unit-linked" contract

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