Information Percolation in Segmented Markets

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Learning in Markets

Informational Role of Prices: Hayek (1945)

- **Centralized Exchanges:**

- **Decentralized Markets:** What if learning happens through local interactions/private negotiations?
Contributions of Today’s Paper

1. tractable model of information diffusion in over-the-counter markets with investor segmentation by preferences, initial information, and connectivity.

2. double auction with common values.

3. effects of information and connectivity on profits:
   - more informed/connected investors attain higher expected profits than less informed/connected investors if they can disguise trades.
   - more informed/connected investors may not attain higher expected profits than less informed/connected investors if characteristics are commonly observed.
Outline of the Talk

1. Information Percolation
2. Segmented Markets
3. Double Auction
4. Connectedness and Information
Model Primitives

Duffie and Manso (2007) and Duffie, Giroux, and Manso (2010):

- Continuum of agents

- Two possible states of nature $Y \in \{0, 1\}$.

- Each agent is initially endowed with signals $S = \{s_1, \ldots, s_n\}$ s.t. $P(s_i = 1 \mid Y = 1) \geq P(s_i = 1 \mid Y = 0)$

- For every pair agents, their initial signals are $Y$-conditionally independent

- Random matching, intensity $\lambda$. 
After observing signals $S = \{s_1, \ldots, s_n\}$, the logarithm of the likelihood ratio between states $Y = 0$ and $Y = 1$ is by Bayes’ rule:

$$
\log \frac{P(Y = 0 \mid s_1, \ldots, s_n)}{P(Y = 1 \mid s_1, \ldots, s_n)} = \log \frac{P(Y = 0)}{P(Y = 1)} + \sum_{i=1}^{n} \log \frac{P(s_i \mid Y = 0)}{P(s_i \mid Y = 1)}.
$$

We say that the “type” $\theta$ associated with this set of signals is

$$
\theta = \sum_{i=1}^{n} \log \frac{P(s_i \mid Y = 0)}{P(s_i \mid Y = 1)}.
$$
What Happens in a Meeting?

- Upon meeting, agents participate in a double auction.

- If bids are strictly increasing in the type associated with the signals agents have collected, then bids reveal type.
Information is Additive in Type Space

**Proposition:** Let $S = \{s_1, \ldots, s_n\}$ and $R = \{r_1, \ldots, r_m\}$ be independent sets of signals, with associated types $\theta$ and $\phi$. If two agents with types $\theta$ and $\phi$ reveal their types to each other, then both agents achieve the posterior type $\theta + \phi$.

This follows from Bayes’ rule, by which

$$
\log \frac{P(Y = 0 \mid S, R, \theta + \phi)}{P(Y = 1 \mid S, R, \theta + \phi)} = \log \frac{P(Y = 0)}{P(Y = 1)} + \theta + \phi,
$$

and

$$
= \log \frac{P(Y = 0 \mid \theta + \phi)}{P(Y = 1 \mid \theta + \phi)}
$$
**Proposition:** Let $S = \{s_1, \ldots, s_n\}$ and $R = \{r_1, \ldots, r_m\}$ be independent sets of signals, with associated types $\theta$ and $\phi$. If two agents with types $\theta$ and $\phi$ reveal their types to each other, then both agents achieve the posterior type $\theta + \phi$.

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$$= \log \frac{P(Y = 0 \mid \theta + \phi)}{P(Y = 1 \mid \theta + \phi)}$$

By induction, this property holds for all subsequent meetings.
Similarity with Gas Kinetics
Solution for Cross-Sectional Distribution of Information

The Boltzmann equation for the cross-sectional distribution $\mu_t$ of types is

$$\frac{d}{dt} \mu_t = -\lambda \mu_t + \lambda \mu_t \ast \mu_t.$$

with a given initial distribution of types $\mu_0$. 
Solution for Cross-Sectional Distribution of Information

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**Proposition:** The unique solution is the Wild sum

$$\mu_t = \sum_{n \geq 1} e^{-\lambda t} \left(1 - e^{-\lambda t}\right)^{n-1} \mu_0^* n.$$
Proof of Wild Summation

Taking the Fourier transform $\hat{\mu}_t$ of $\mu_t$ of the Boltzmann equation

$$\frac{d}{dt} \mu_t = -\lambda \mu_t + \lambda \mu_t \ast \mu_t,$$

we obtain the following ODE

$$\frac{d}{dt} \hat{\mu}_t = -\lambda \hat{\mu}_t + \lambda \hat{\mu}_t^2,$$

whose solution is

$$\hat{\mu}_t = \hat{\mu}_0 \frac{e^{\lambda t}(1 - \hat{\mu}_0) + \hat{\mu}_0}{e^{\lambda t} - \hat{\mu}_0}.$$

This solution can be expanded as

$$\hat{\mu}_t = \sum_{n \geq 1} e^{-\lambda t} (1 - e^{-\lambda t})^{n-1} \hat{\mu}_0^n,$$

which is the Fourier transform of the Wild sum.
Multi-Agent Meetings

The Boltzmann equation for the cross-sectional distribution $\mu_t$ of types is

$$\frac{d}{dt} \mu_t = -\lambda \mu_t + \lambda \mu_t^m.$$ 

Taking the Fourier transform, we obtain the ODE,

$$\frac{d}{dt} \hat{\mu}_t = -\lambda \hat{\mu}_t + \lambda \hat{\mu}_t^m,$$

whose solution satisfies

$$\hat{\mu}_t^{m-1} = \frac{\hat{\mu}_0^{m-1}}{e^{(m-1)\lambda t}(1 - \hat{\mu}_0^{m-1}) + \hat{\mu}_0^{m-1}}.$$  \hspace{1cm} (1)
Groups of 2 (blue) versus Groups of 3 (red)
Groups of 2 (blue) versus Groups of 3 (red)
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New Private Information

Suppose that, independently across agents as above, each agent receives, at Poisson mean arrival rate $\rho$, a new private set of signals whose type outcome $y$ is distributed according to a probability measure $\nu$. Then the evolution equation is extended to

$$\frac{d}{dt} \mu_t = -(\lambda + \rho) \mu_t + \lambda \mu_t \ast \mu_t + \rho \mu_t \ast \nu.$$ 

Taking Fourier transforms, we obtain the following ODE

$$\frac{d}{dt} \hat{\mu}_t = -(\lambda + \rho) \hat{\mu}_t + \lambda \hat{\mu}_t^2 + \rho \hat{\mu}_t \hat{\nu}.$$ 

whose solution satisfies

$$\hat{\mu}_t = \frac{\hat{\mu}_0}{e^{(\lambda+\rho(1-\hat{\nu}))t} (1 - \hat{\mu}_0) + \hat{\mu}_0}$$
Other Extensions

- Public information releases
  - Duffie, Malamud, and Manso (2010).

- Endogenous search intensity
1 At public information release random times \( \{T_1, T_2, \ldots \} \) (Poisson arrival process with intensity \( \eta \)) \( n \) randomly selected agents have their posterior probabilities revealed to all agents.

2 We allow for random number of agents in each meeting and in each public information release:
   - Meeting group size \( m \): \( q_l = P(m = l) \).
   - Public information release group size \( n \): \( p_k = P(n = k) \).
Evolution of type distribution

**Theorem.** Given the variable $X$ of common concern, the probability distribution of each agent’s type at time $t$ is $\nu_t = \alpha_t * \beta_t$, where $\alpha_t = h(\mu_0, t)$ is the type distribution in a model with no public releases of information, satisfying the differential equation

$$
\frac{d\alpha_t}{dt} = \lambda \left( \sum_{l=2}^{\infty} q_l \alpha_t^l - \alpha_t \right), \quad \alpha_0 = \mu_0,
$$

(2)

and where $\beta_t$ is the probability distribution over types that solves the differential equation

$$
\frac{d\beta_t}{dt} = -\eta \beta_t + \eta \beta_t * \sum_{k=1}^{\infty} p_k \alpha_t^k,
$$

(3)

with initial condition given by the Dirac measure $\delta_0$ at zero.
The rate of Convergence

Let

\[ s \mapsto M(s) = \int e^{sx} \, d\mu_0(x) \]

and

\[ R = \sup_{y \in \mathbb{R}} \left( -\log M(y) \right). \tag{4} \]

and

\[ \Phi(z) = \sum_{n=1}^{\infty} p_n z^n, \]

**Theorem** Convergence is exponential at the rate \( \lambda + \eta \), as long as \( \lambda > 0 \). Otherwise, the rate

\[ \rho = \eta \left( 1 - \Phi(e^{-R}) \right). \tag{5} \]

is strictly less than \( \eta \).
Same as the previous model except that:

- $N$ classes of investors.
- Agent of class $i$ has matching intensity $\lambda_i$.
- Upon meeting, the probability that a class-$j$ agent is selected as a counterparty is $\kappa_{i,j}$. 
Evolution of Type Distribution

The evolution equation is given by:

\[
\frac{d}{dt} \psi_{it} = -\lambda_i \psi_{it} + \lambda_i \psi_{it} \sum_{j=1}^{N} \kappa_{ij} \psi_{jt}, \quad i \in \{1, \ldots, N\}.
\]

Taking Fourier transforms we obtain:

\[
\frac{d}{dt} \hat{\psi}_{it} = -\lambda_i \hat{\psi}_{it} + \lambda_i \hat{\psi}_{it} \sum_{j=1}^{N} \kappa_{ij} \hat{\psi}_{jt}, \quad i \in \{1, \ldots, N\},
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Special Case: $N = 2$ and $\lambda_1 = \lambda_2$

**Proposition:** Suppose $N = 2$ and $\lambda_1 = \lambda_2 = \lambda$. Then

\[
\hat{\psi}_1 = \frac{e^{-\lambda t} \left( \hat{\psi}_{20} - \hat{\psi}_{10} \right)}{\hat{\psi}_{20} e^{-\hat{\psi}_{20}(1-e^{-\lambda t})} - \hat{\psi}_{10} e^{-\hat{\psi}_{10}(1-e^{-\lambda t})}} \hat{\psi}_{10} e^{-\hat{\psi}_{10}(1-e^{-\lambda t})}
\]

\[
\hat{\psi}_2 = \frac{e^{-\lambda t} \left( \hat{\psi}_{20} - \hat{\psi}_{10} \right)}{\hat{\psi}_{20} e^{-\hat{\psi}_{20}(1-e^{-\lambda t})} - \hat{\psi}_{10} e^{-\hat{\psi}_{10}(1-e^{-\lambda t})}} \hat{\psi}_{20} e^{-\hat{\psi}_{20}(1-e^{-\lambda t})}.
\]
General Case: Wild Sum Representation

**Theorem:** There is a unique solution of the evolution equation, given by

$$
\psi_{it} = \sum_{k \in \mathbb{Z}_+^N} a_{it}(k) \psi^{*k_1}_{10} \ast \cdots \ast \psi^{*k_N}_{N0},
$$

where $\psi^{*n}_{i0}$ denotes $n$-fold convolution,

$$
a'_{it} = -\lambda_i a_{it} + \lambda_i a_{it} \ast \sum_{j=1}^N \kappa_{ij} a_{jt}, \quad a_{i0} = \delta_{e_i},
$$

$$
(a_{it} \ast a_{jt})(k_1, \ldots, k_N) = \sum_{l=(l_1, \ldots, l_N) \in \mathbb{Z}_+^N, l < k} a_{it}(l) a_{jt}(k - l),
$$

and

$$
a_{it}(e_i) = e^{-\lambda_{i}t} a_{i0}(e_i).
$$
Double Auction

- At some time $T$, the economy ends and the utility realized by an agent of class $i$ for each additional unit of the asset is

$$U_i = v_i Y + v^H (1 - Y),$$

measured in units of consumption, for strictly positive constants $v^H$ and $v_i < v^H$, where $Y$ is a non-degenerate 0-or-1 random variable whose outcome will be revealed at time $T$.

- If $v_i = v_j$, no trade (Milgrom and Stokey (1982)), so that $\kappa_{ij} = 0$.

- Meeting between two agents $v_i > v_j$, then $i$ is buyer and $j$ is seller.

- Upon meeting, participate in a double auction. If the buyer’s bid $\beta$ is higher than the seller’s ask $\sigma$, trade occurs at the price $\sigma$. 

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Equilibrium

The prices \((\sigma, \beta)\) constitute an equilibrium for a seller of class \(i\) and a buyer of class \(j\) provided that, fixing \(\beta\), the offer \(\sigma\) maximizes the seller’s conditional expected gain,

\[
E \left[ (\sigma - E(U_i | \mathcal{F}_S \cup \{\beta\}))1_{\{\sigma<\beta\} | \mathcal{F}_S} \right],
\]

and fixing \(\sigma\), the bid \(\beta\) maximizes the buyer’s conditional expected gain

\[
E \left[ (E(U_j | \mathcal{F}_B \cup \{\sigma\}) - \sigma)1_{\{\sigma<\beta\} | \mathcal{F}_B} \right].
\]

Counterexample: Reny and Perry (2006)
Equilibrium

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\]

Counterexample: Reny and Perry (2006)
Restriction on the Initial Information Endowment

Lemma: Suppose that each signal $Z$ satisfies

$$P(Z = 1 | Y = 0) + P(Z = 1 | Y = 1) = 1.$$ 

Then, for each agent class $i$ and time $t$, the type density $\psi_{it}$ satisfies

$$\psi_H^{it}(x) = e^x \psi_H^{it}(-x), \quad \psi^L_{it}(x) = \psi_H^{it}(-x) \quad x \in \mathbb{R}. $$

and the hazard rate condition

$$h_H^{it}(x) \overset{\text{def}}{=} \frac{\psi_H^{it}(x)}{\int_x^{+\infty} \psi_H^{it}(y) \, dy} \geq \frac{\psi_L^{it}(x)}{\int_x^{+\infty} \psi_L^{it}(y) \, dy} \overset{\text{def}}{=} h_L^{it}(x).$$
Lemma: For any $V_0 \in \mathbb{R}$, there exists a unique solution $V_2(\cdot)$ on $[v_i, v^H]$ to the ODE

$$V'_2(z) = \frac{1}{v_i - v_j} \left( \frac{z - v_i}{v^H - z} \frac{1}{h^H_{it}(V_2(z))} + \frac{1}{h^L_{it}(V_2(z))} \right), \quad V_2(v_i) = V_0.$$ 

This solution, also denoted $V_2(V_0, z)$, is monotone increasing in both $z$ and $V_0$. Further, $\lim_{v \to v^H} V_2(v) = +\infty$. The limit $V_2(-\infty, z) = \lim_{V_0 \to -\infty} V_2(V_0, z)$ exists. Moreover, $V_2(-\infty, z)$ is continuously differentiable with respect to $z$. 
**Bidding Strategies**

**Proposition:** Suppose that \((S, B)\) is a continuous equilibrium such that \(S(\theta) \leq v^H\) for all \(\theta \in \mathbb{R}\). Let \(V_0 = B^{-1}(v_i) \geq -\infty\). Then,

\[
B(\phi) = V_2^{-1}(\phi), \quad \phi > V_0,
\]

Further, \(S(-\infty) = \lim_{\theta \to -\infty} S(\theta) = v_i\) and \(S(+\infty) = \lim_{\theta \to -\infty} S(\theta) = v^H\), and for any \(\theta\), we have \(S(\theta) = V_1^{-1}(\theta)\) where

\[
V_1(z) = \log \frac{z - v_i}{v^H - z} - V_2(z), \quad z \in (v_i, v^H).
\]

Any buyer of type \(\phi < V_0\) will not trade, and has a bidding policy \(B\) that is not uniquely determined at types below \(V_0\).
**Tail Condition**

**Definition:** We say that a probability density \( g(\cdot) \) on the real line is of exponential type \( \alpha \) at \( +\infty \) if, for some constants \( c > 0 \) and \( \gamma > -1 \),

\[
\lim_{x \to +\infty} \frac{g(x)}{x^\gamma e^{\alpha x}} = c
\]

In this case, we write \( g(x) \sim \text{Exp}_{+\infty}(c, \gamma, \alpha) \).
Suppose $N = 1$, and let $\lambda = \lambda_1$ and $\psi_t = \psi_{1t}$. The Laplace transform $\hat{\psi}_t$ of $\psi_t$ is given by

$$\hat{\psi}_t(z) = \frac{e^{-\lambda t} \hat{\psi}_0(z)}{1 - (1 - e^{-\lambda t}) \hat{\psi}_0(z)}$$

and $\psi_t(x) \sim \text{Exp}_{+\infty}(c_t, 0, -\alpha_t)$ in $t$, where $\alpha_t$ is the unique positive number $z$ solving

$$\hat{\psi}_0(z) = \frac{1}{1 - e^{-\lambda t}},$$

and where

$$c_t = \frac{e^{-\lambda t}}{(1 - e^{-\lambda t})^2} \frac{d}{dz} \hat{\psi}_0(\alpha_t).$$

Furthermore, $\alpha_t$ is monotone decreasing in $t$, with $\lim_{t \to \infty} \alpha_t = 0$. 

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Proposition: Suppose that, for all $t$ in $[0, T]$, there are $\alpha_i(t)$, $c_i(t)$, and $\gamma_i(t)$ such that

$$\psi_{it}^H(x) \sim \text{Exp}_{+\infty}(c_i(t), \gamma_i(t), -\alpha_i(t)).$$

If $\alpha_i(T) < 1$, then there is no equilibrium associated with $V_0 = -\infty$. Moreover, if $v_i - v_j$ is sufficiently large and if $\alpha_i(T) > \alpha^*$, where $\alpha^*$ is the unique positive solution to $\alpha^* = 1 + 1/(\alpha^*2\alpha^*)$ (which is approximately 1.31), then there exists a unique strictly monotone equilibrium associated with $V_0 = -\infty$. This equilibrium is in undominated strategies, and maximizes total welfare among all continuous equilibria.
Class-$i$ Agent Utility

The expected future profit at time $t$ of a class-$i$ agent is

$$\mathcal{U}_i(t, \Theta_t) = E \left[ \sum_{\tau_k > t} \sum_j \kappa_{ij} \pi_{ij}(\tau_k, \Theta_{\tau_k}) \bigg| \Theta_t \right],$$

where $\tau_k$ is this agent’s $k$-th auction time and $\pi_{ij}(t, \theta)$ is the expected profit of a class-$i$ agent of type $\theta$ entering an auction at time $t$ with a class-$j$ agent.

Agents may be able to disguise the characteristics determining their information at a particular auction. In this case, we denote the expected future profit at time $t$ of a class-$i$ agent as $\hat{\mathcal{U}}_i(t, \Theta_t)$. 
The Value of Initial Information and Connectivity When Trades Can be Disguised

**Theorem:** Suppose that \( v_1 = v_2 \). If \( \lambda_2 \geq \lambda_1 \) and if the initial type densities \( \psi_{10} \) and \( \psi_{20} \) are distinguished by the fact that the density \( p_2 \) of the number of signals received by class-2 agents has first-order stochastic dominance over the density \( p_1 \) of the number of signals by class-1 agents, then

\[
\frac{E[\hat{U}_2(t, \Theta_{2t})]}{\lambda_2} \geq \frac{E[\hat{U}_1(t, \Theta_{1t})]}{\lambda_1}, \quad t \in [0, T].
\]

The above inequality holds strictly if, in addition, \( \lambda_2 > \lambda_1 \) or if \( p_2 \) has strict dominance over \( p_1 \).
What if Characteristics are Commonly Observed?

- trade-off between adverse selection and gains from trade.

- more informed/connected investor may achieve lower profits than less informed/connected investor.

- If $v_1 = v_2 = 0.9$, $v_3 = 0$, $v^H = 1.9$,

$$
\psi_{10}(x) = 12 \frac{e^{3x}}{(1 + e^x)^5},
$$

and $\psi_{20}(x) = \psi_{10} \ast \psi_{10}$.

Then,

$$
E[U_2(t, \Theta_{1t})] < E[U_1(t, \Theta_{2t})]
$$

and

$$
E[\hat{U}_1(t, \Theta_{1t})] < E[U_1(t, \Theta_{2t})].
$$
What if Characteristics are Commonly Observed?
Even If Characteristics are Commonly Observed
Connectivity May be Valuable

Proposition: Suppose that \( \kappa_1 = \kappa_2 \), \( v_1 = v_2 \) and \( \lambda_1 < \lambda_2 \), and suppose that class-1 and class-2 investors have the same initial information quality, that is, \( \psi_{10} = \psi_{20} \), and assume the exponential tail condition
\[
\psi_{it}^H \sim \text{Exp}_{+\infty} (c_{it}, \gamma_{it}, -\alpha_{it})
\]
for all \( i \) and \( t \), with \( \alpha_{10} > 3 \), \( \alpha_{30} < 3 \) and
\[
\alpha_{30} > \frac{\alpha_{10} - 1}{3 - \alpha_{10}},
\]
and
\[
\frac{\alpha_{1t} + 1}{\alpha_{1t} - 1} > \alpha_{3t}, \quad t \in [0, T].
\]

If \( \frac{v_1 - v_3}{v^H - v_1} \) is sufficiently large, then for any time \( t \) we have
\[
\frac{E[U_2(t, \Theta_{2t})]}{\lambda_2} > \frac{E[\hat{U}_2(t, \Theta_{2t})]}{\lambda_2} > \frac{E[\hat{U}_1(t, \Theta_{1t})]}{\lambda_1} > \frac{E[U_1(t, \Theta_{1t})]}{\lambda_1}.
\]
Subsidizing Order Flow

- Investors $i$ and $j$ with $v_i = v_j$ meet at time $t$.
- Enter a swap agreement by which the amount
  \[ k \left[ (p_j(t) - Y)^2 - (p_i(t) - Y)^2 \right], \]
  will be paid by investor $i$ to investor $j$ at time $T$.
- Increase connectivity of class $i$ investors.
- When would investors want to subsidize order flow?
Concluding Remarks

- tractable model of information diffusion in over-the-counter markets.

- initial information and connectivity may or may not increase profits:
  - more informed/connected investors attain higher profits than less informed connected investors when investors can disguise trades.
  - more informed/connected investors may attain lower profits than less informed connected investors when investors’ characteristics are commonly observed.
Other Applications

- centralized exchanges, decentralized information transmission
- bank runs
- knowledge spillovers
- social learning
- technology diffusion
Thank You!