Co-integration in continuous-time with applications in finance

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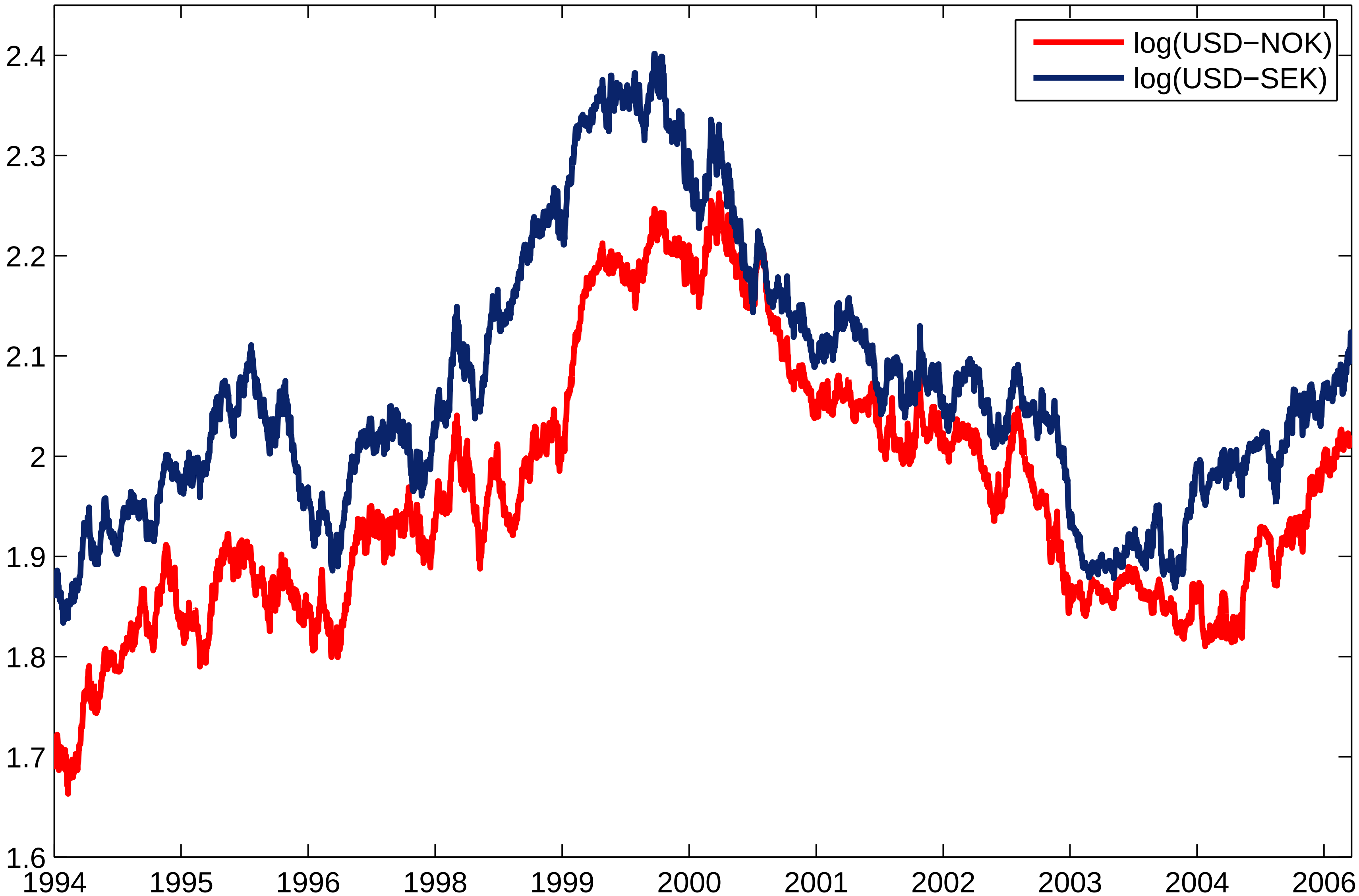
Paper: Time series regression on integrated continuous-time processes with heavy and light tails

Motivation

Exchange Rates: USD-NOK-SEK



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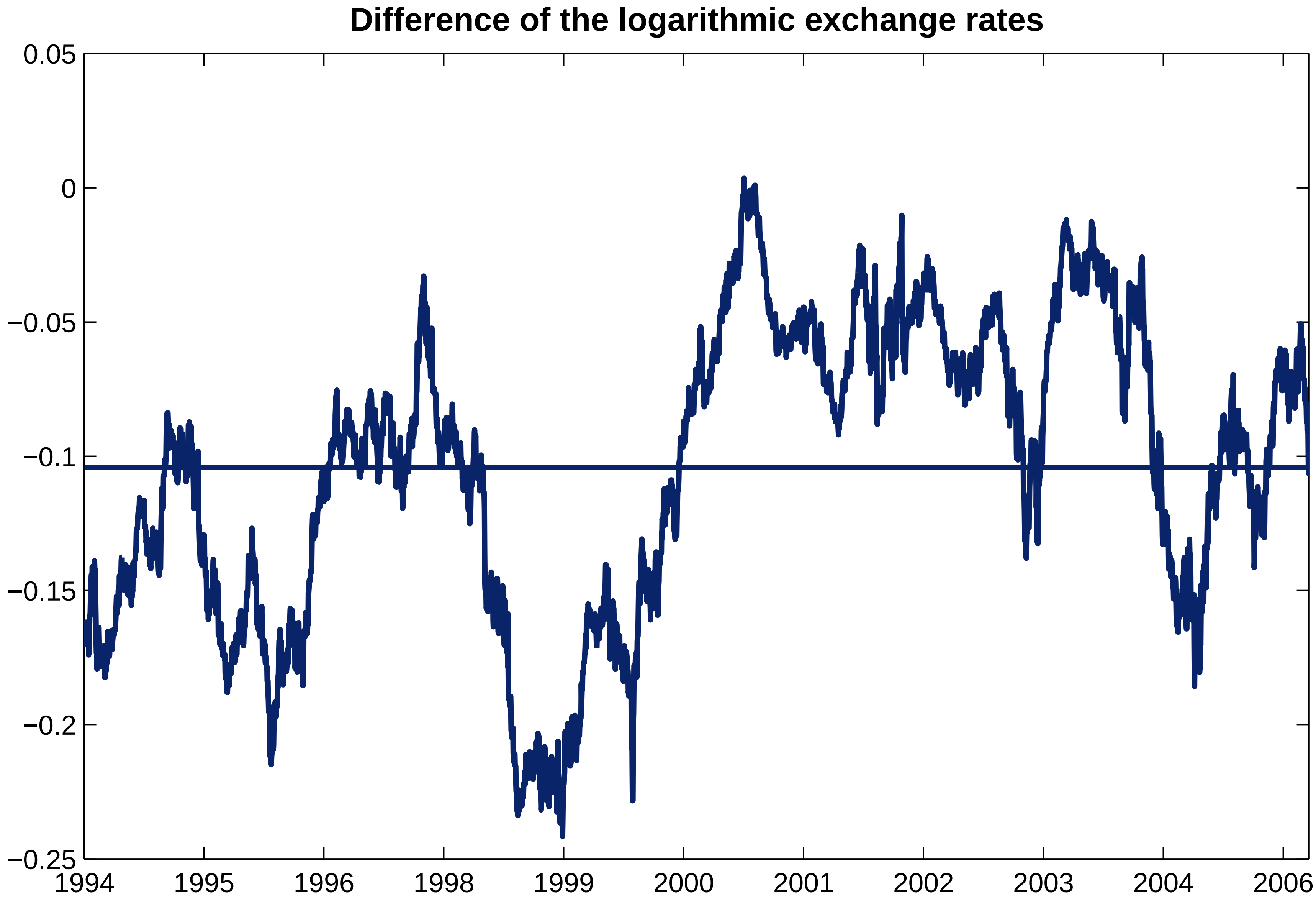

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Example: Lévy process

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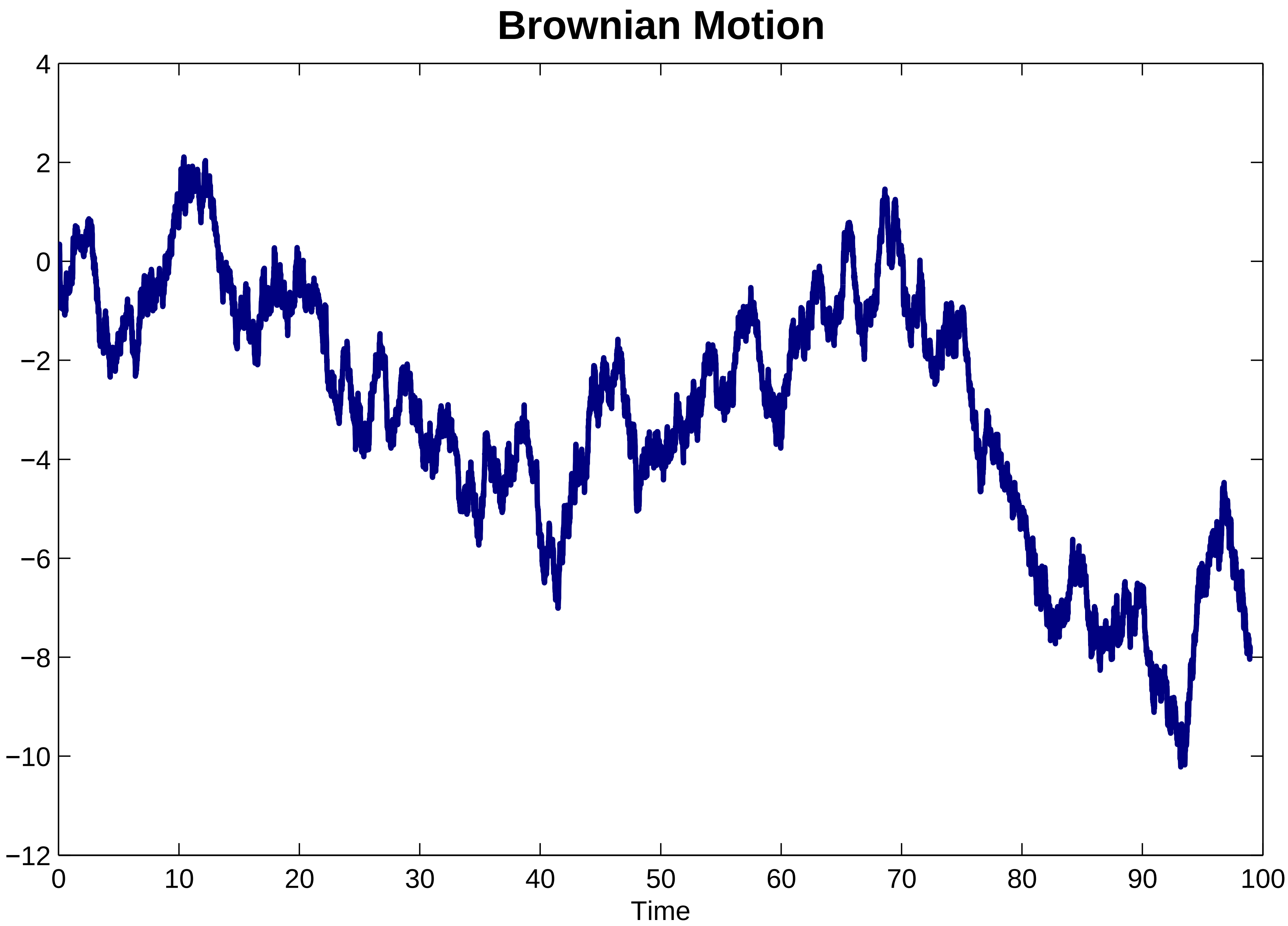
Example: Brownian motion, compound Poisson process

Brownian Motion



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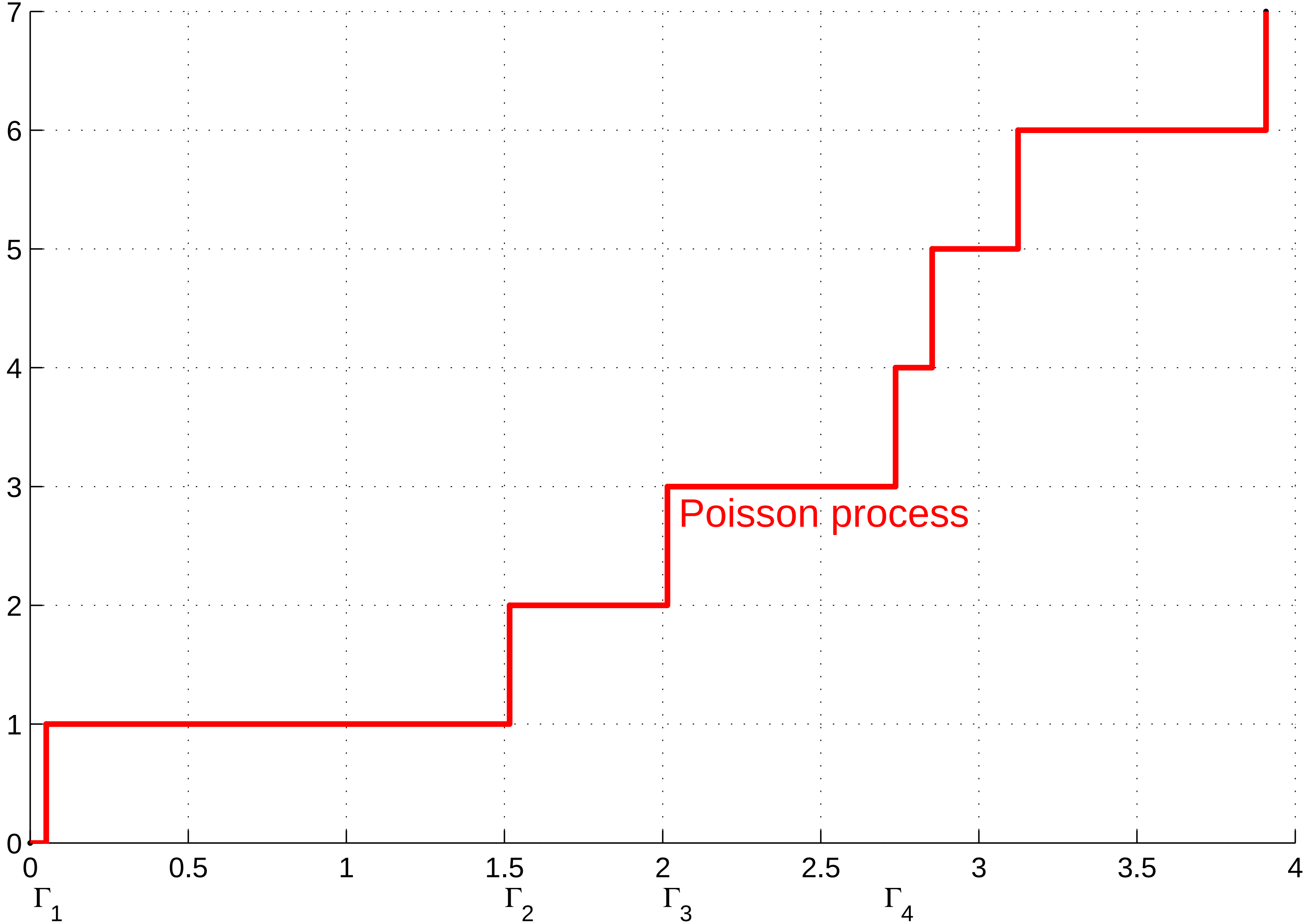

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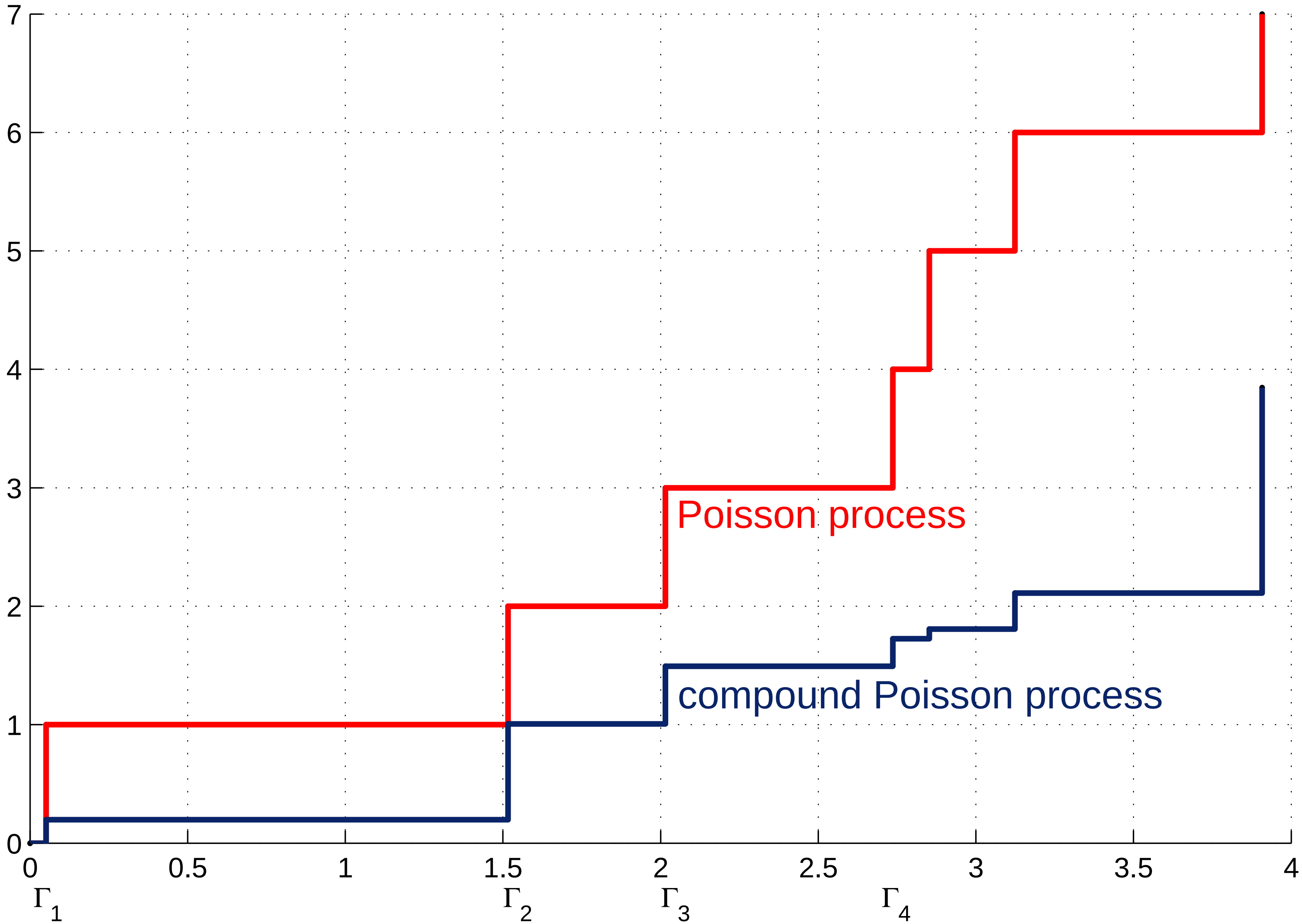

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References: – Granger (1981) [Nobel Prize (2003)]

– Engle and Granger (1987)

Applications of Co-integration in Finance

Empirical evidence of co-integrated financial time series:

- foreign currency spot and future markets
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- interest rates in different countries
- equity markets in different countries
- stock prices within a given industry
- property-liability underwriting margins and interest rates

Asset Model

A model for the price of an asset $\boldsymbol{S = (S(t))_{t \geq 0}}$ is

$$\boldsymbol{S(t) = \exp(L(t))} \quad \text{for } \boldsymbol{t \geq 0}$$

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Example: geometric Brownian motion

$$S(t) = S_0 \exp(\sigma B(t) - (\mu - \sigma^2/2)t) \quad \text{for } t \geq 0$$

where $(B(t))_{t \geq 0}$ is a Brownian motion.

Asset Model

- The **logarithmic** price of asset 1 is

$$\boldsymbol{Y(t) := \log S_1(t) = L_1(t) \quad \text{for } t \geq 0}$$

where $(L_1(t))_{t\geq 0}$ is a Lévy process.

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- The **logarithmic** price of asset 2 is

$$\boldsymbol{X(t) := \log S_2(t) = AY(t) + Z(t)} \quad \text{for } \boldsymbol{t \geq 0}$$

and some $A \in \mathbb{R}$.

Spread

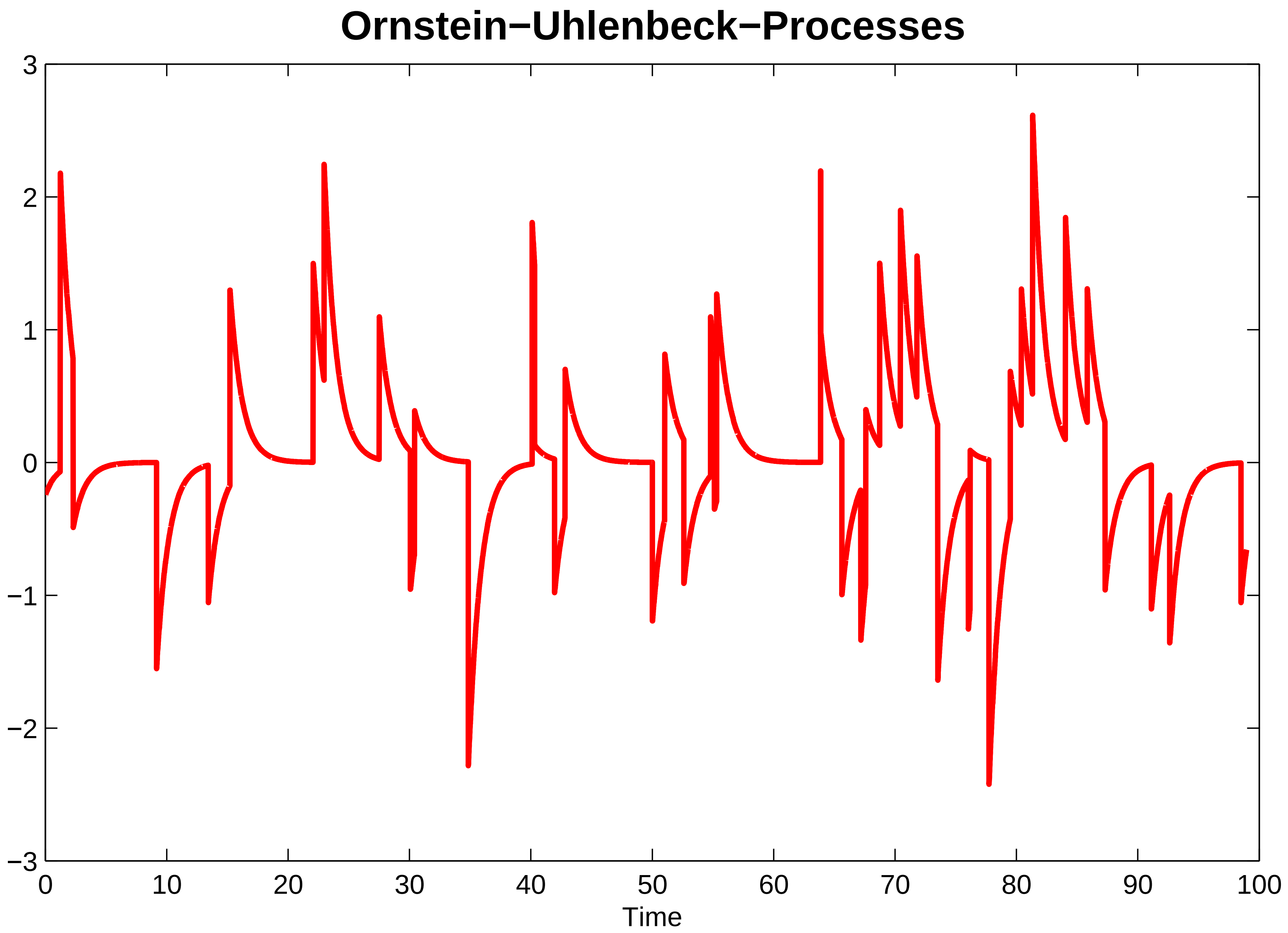
The spread $(Z(t))_{t\geq 0}$ is an Ornstein-Uhlenbeck process

$$Z(t) = \mathrm{e}^{-\lambda t} Z(0) + \int_0^t \mathrm{e}^{-\lambda(t-s)}\, \mathrm{d}L_2(s) \quad \text{for } t \geq 0$$

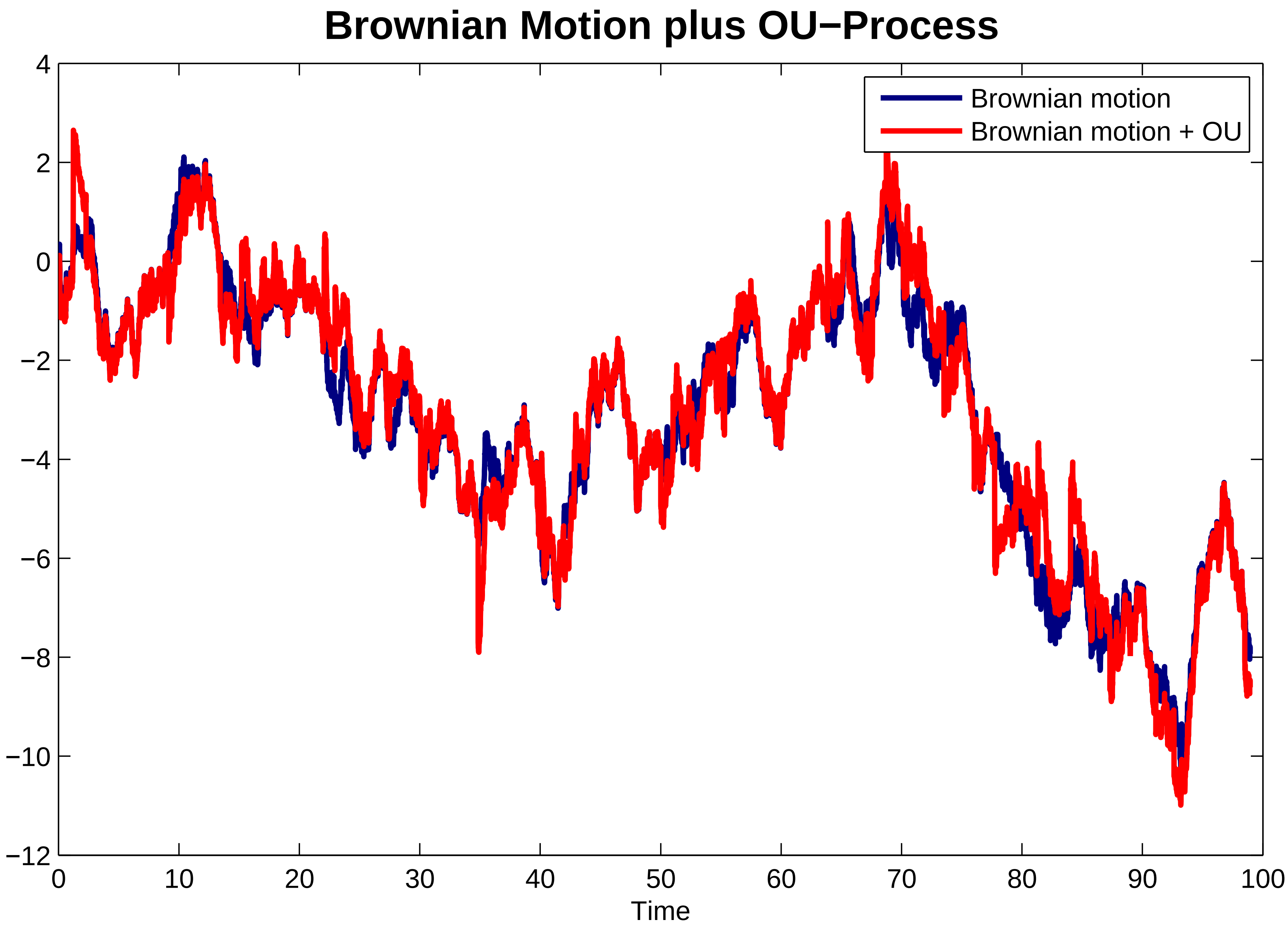
where

- $(L_2(t))_{t\geq 0}$ is a Lévy process,
- $\lambda > 0$.

Ornstein-Uhlenbeck Process



Brownian Motion plus Ornstein-Uhlenbeck Process



Asset Model

Asset model,

$$S(t) = \exp(L(t) + \zeta(t)) \quad \text{for } t \geq 0$$

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- $(L(t))_{t\geq 0}$ Lévy process
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Note: Pairs trading uses these co-integrated models

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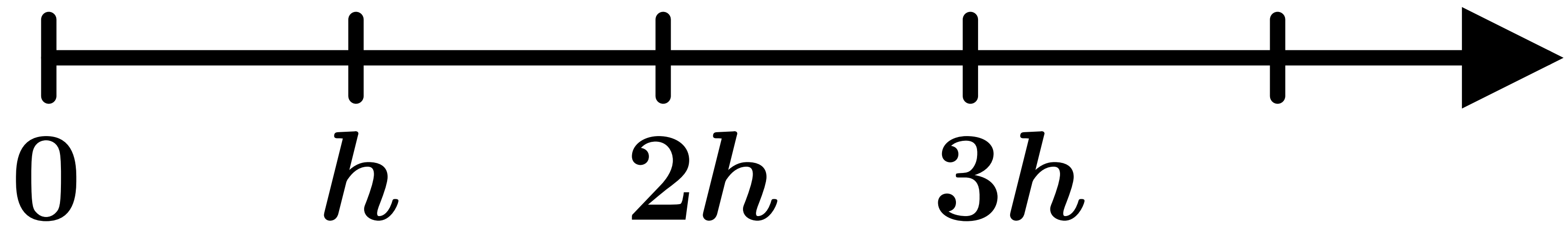
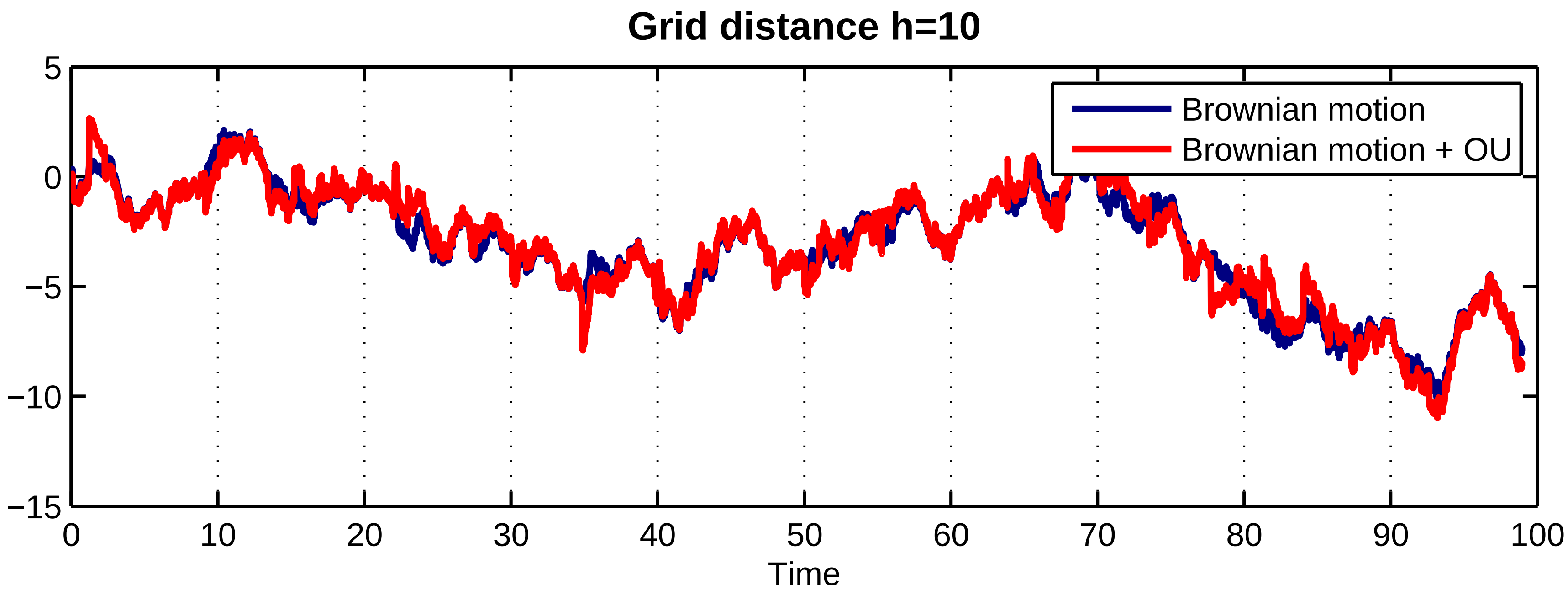
Estimators

Observation Scheme





Notation

Let $h > 0$.

$$\mathbb{X}'_n = \ (X(h), \ldots, X(nh)) \ \in \mathbb{R}^n$$
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Then

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Least squares estimator for A:

$$\widehat{\boldsymbol{A}}_{\boldsymbol{n}} = \mathbb{X}'_{\boldsymbol{n}}\mathbb{Y}_{\boldsymbol{n}}(\mathbb{Y}'_{\boldsymbol{n}}\mathbb{Y}_{\boldsymbol{n}})^{-1}.$$

Assumptions

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Co-integrated model:

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$$\boldsymbol{n}\mathbb{P}(\boldsymbol{a}_n^{-1}\boldsymbol{W} \in \cdot) \xrightarrow{\nu} \boldsymbol{\mu}(\cdot) \text{ on } \boldsymbol{\mathcal{B}}(\overline{\mathbb{R}}\backslash\{\mathbf{0}\})$$

and for some $C_1, C_2 \geq 0$, $C_1 + C_2 > 0$,

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$\Longrightarrow \lim_{x\to\infty} \frac{\mathbb{P}(|W|>xt)}{\mathbb{P}(|W|>x)} = t^{-\alpha}$ $\Longrightarrow$ heavy tailed distributions

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- $$\begin{aligned} \mathbb{P}(\boldsymbol{W} > x) &\sim C_1 x^{-\alpha} \quad \text{as } x \to \infty, \\ \mathbb{P}(-\boldsymbol{W} > x) &\sim C_2 x^{-\alpha} \quad \text{as } x \to \infty. \end{aligned}$$

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Note: If $\alpha < 2$, then $\mathbb{E}(W^2) = \infty$.

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- $W_1 \in \mathcal{R}_{-\alpha}(a_n, \mu)$ with $\alpha \in (0, 2)$. Then

$$\frac{1}{a_n} \sum_{k=1}^{n} W_k \stackrel{n\to\infty}{\Longrightarrow} S_\alpha$$

where S_α is an α-stable random variable with Lévy measure μ.

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  holds.
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Under some additional assumption L_1 and L_2 can also be dependent.

Asymptotics of $\widehat{A}_n$

Limit Behavior

Then as $n \to \infty$,

$$\boldsymbol{n a_{nh} b_{nh}^{-1} K_{\lambda h, \alpha_2} (\widehat{A}_n - A)}$$

where

- $K_{\lambda h, \alpha_2} = (\alpha_2 \lambda h)^{-\frac{1}{\alpha_2}} (1 - \mathrm{e}^{-\alpha_2 \lambda h})^{-\frac{1}{\alpha_2}} (1 - \mathrm{e}^{-\lambda h})$,

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where

- $K_{\lambda h,\alpha_2} = (\alpha_2 \lambda h)^{-\frac{1}{\alpha_2}} (1 - \mathrm{e}^{-\alpha_2 \lambda h})^{-\frac{1}{\alpha_2}} (1 - \mathrm{e}^{-\lambda h})$,
- $\alpha_i < 2$: $(S_i(t))_{t \geq 0}$ is an α_i-stable Lévy process with Lévy measure μ_i,
- $\alpha_i = 2$: $(S_i(t))_{t \geq 0}$ is a Brownian motion,
- $(S_1(t))_{t \geq 0}$ and $(S_2(t))_{t \geq 0}$ are independent.

Limit Behavior

Then as $n \to \infty$,

$$\boldsymbol{n a_{nh} b_{nh}^{-1} K_{\lambda h,\alpha_2}(\widehat{A}_n - A)} \Longrightarrow \left(\int_0^1 \mathrm{S}_1(s-)\mathrm{dS}_2(s)\right)\left(\int_0^1 \mathrm{S}_1(s)^2\mathrm{d}s\right)^{-1}.$$

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In particular,

$$\widehat{\boldsymbol{A}}_n \xrightarrow{\mathbb{P}} \boldsymbol{A} \quad \text{as } \boldsymbol{n} \to \infty$$

if $\alpha_2 > \alpha_1/(\alpha_1 + 1)$, i.e. $\widehat{A}_n$ is a consistent estimator.

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Examples:

- $(S_2(t))_{t \geq 0}$ Brownian motion
- $\mathbb{P}(|L_1(1)| > x) \sim \mathbb{P}(|L_2(1)| > x)$ as $x \to \infty$ $\quad \Longrightarrow \quad \alpha_1 = \alpha_2$

Special case

- On the one hand,

$$\lim_{u\to\infty}\frac{\mathbb{P}(L_1(1)>u)}{\mathbb{P}(|L_1(1)|>u)}=\lim_{u\to\infty}\frac{\mathbb{P}(-L_1(1)>u)}{\mathbb{P}(|L_1(1)|>u)}=\frac{1}{2},$$

- and on the other hand,

$$\lim_{u\to\infty}\frac{\mathbb{P}(L_2(1)>u)}{\mathbb{P}(|L_2(1)|>u)}=\lim_{u\to\infty}\frac{\mathbb{P}(-L_2(1)>u)}{\mathbb{P}(|L_2(1)|>u)}=\frac{1}{2}.$$

Central Limit Result

Then

$$n a_{nh} b_{nh}^{-1} K_{\lambda h, \alpha_2} (\widehat{A}_n - A)$$

$$\Longrightarrow \left(\int_0^1 S_1^*(s-) \mathrm{d} S_2^*(s) \right) \left(\int_0^1 S_1^*(s)^2 \mathrm{d} s \right)^{-1} =: G^*$$

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where

- $(S_1^*(t))_{t \geq 0}$ is a $S_{\alpha_1}(1, 0, 0)$-stable Lévy process,
- $(S_2^*(t))_{t \geq 0}$ is a $S_{\alpha_2}(1, 0, 0)$-stable Lévy process,
- $(S_1^*(t))_{t \geq 0}$ and $(S_2^*(t))_{t \geq 0}$ are independent.

Confidence Intervals

The asymptotic $(1-p)$-confidence interval of A is

$$\left[\widehat{A}_n \pm \frac{x_p(\alpha_1,\alpha_2)b_{nh}}{na_{nh}}(\alpha_2\lambda h)^{-\frac{1}{\alpha_2}}(1-\mathrm{e}^{-\alpha_2\lambda h})^{\frac{1}{\alpha_2}}(1-\mathrm{e}^{-\lambda h})^{-1}\right]$$

where $x_p(\alpha_1,\alpha_2)$ is the $(1-\frac{p}{2})$-quantile of G^*.

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- $\mathbb{E}L_1(1)^2 = 1$, $\mathbb{E}L_2(1)^2 = 1$ and $\alpha = 2$.

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Confidence Intervals

$d=1$	$\alpha=0.9$	$\alpha=1.2$	$\alpha=1.4$	$\alpha=1.5$	$\alpha=1.6$	$\alpha=1.8$	$\alpha=2$
$p=0.1$	12.5	7.0	5.5	5.1	4.8	4.2	3.8
$p=0.05$	27.9	12.5	8.9	7.8	7.1	5.7	5.0
$p=0.025$	60.8	22.1	14.3	11.9	10.2	7.6	6.2
$p=0.01$	169.5	45.0	26.0	20.0	16.6	11.0	7.9

Table 1: $x_p(\alpha)$ the $(1-\frac{p}{2})$-quantile of G^*

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$$\left[\widehat{A}_n \pm \frac{x_p(\alpha)}{n} \underbrace{(\alpha\lambda h)^{-\frac{1}{\alpha}}(1-\mathrm{e}^{-\alpha\lambda h})^{\frac{1}{\alpha}}}_{\approx 1}(1-\mathrm{e}^{-\lambda h})^{-1}\right]$$

where $x_p(\alpha)$ is the $(1-\frac{p}{2})$-quantile of G^*.

t-Ratio Test

t-Ratio Test

- Then as $n \to \infty$,

$$
\begin{aligned}
&\boldsymbol{K_{\lambda h} t_{\widehat{A}_n}} \\
&= \boldsymbol{K_{\lambda h}(n^{-1}(\mathbb{X}_n' - \widehat{A}_n \mathbb{Y}_n')(\mathbb{X}_n' - \widehat{A}_n \mathbb{Y}_n')' \mathbb{Y}_n' \mathbb{Y}_n)^{-\frac{1}{2}}(\widehat{A}_n - A)}
\end{aligned}
$$

where $K_{\lambda h} = (1 - \mathrm{e}^{-\lambda h})(1 - \mathrm{e}^{-2\lambda h})^{-\frac{1}{2}} \in \mathbb{R}_+$.

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where $K_{\lambda h} = (1 - \mathrm{e}^{-\lambda h})(1 - \mathrm{e}^{-2\lambda h})^{-\frac{1}{2}} \in \mathbb{R}_+$.

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- Then as $n \to \infty$,

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$$= \boldsymbol{K_{\lambda h}(n^{-1}(\mathbb{X}_n' - \widehat{A}_n \mathbb{Y}_n')(\mathbb{X}_n' - \widehat{A}_n \mathbb{Y}_n')' \mathbb{Y}_n' \mathbb{Y}_n)^{-\frac{1}{2}}(\widehat{A}_n - A)}$$

$$\Longrightarrow \boldsymbol{[S_2^*, S_2^*]_1^{-\frac{1}{2}} \left(\int_0^1 S_1^*(s-) \mathrm{d} S_2^*(s) \right) \left(\int_0^1 S_1^*(s)^2 \mathrm{d}s \right)^{-\frac{1}{2}} =: G^{**}}$$

where $K_{\lambda h} = (1 - \mathrm{e}^{-\lambda h})(1 - \mathrm{e}^{-2\lambda h})^{-\frac{1}{2}} \in \mathbb{R}_+$.

Then A has the $(1 - p)$-confidence interval

$$\left[\widehat{A}_n \pm x_p(\alpha_1, \alpha_2)(1 - \mathrm{e}^{-\lambda h})^{-1}(1 - \mathrm{e}^{-2\lambda h})^{\frac{1}{2}}(n^{-1}(\mathbb{X}_n' - \widehat{A}_n \mathbb{Y}_n')(\mathbb{X}_n' - \widehat{A}_n \mathbb{Y}_n')' \mathbb{Y}_n' \mathbb{Y}_n)^{-\frac{1}{2}} \right]$$

where $x_p(\alpha_1, \alpha_2)$ is the $(1 - \frac{p}{2})$-quantile of G^{**}.

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$p = 10\%$	1.6
$p = 5\%$	1.9
$p = 2.5\%$	2.2
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normal quantiles

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normal quantiles

- Then A has the asymptotic $(1 - p)$-confidence interval

$$\left[\widehat{A}_n \pm x_p(\alpha)(1 - \mathrm{e}^{-\lambda h})^{-1}(1 - \mathrm{e}^{-2\lambda h})^{\frac{1}{2}}(n^{-1}(\mathbb{X}_n' - \widehat{A}_n \mathbb{Y}_n')(\mathbb{X}_n' - \widehat{A}_n \mathbb{Y}_n')'\mathbb{Y}_n'\mathbb{Y}_n)^{-\frac{1}{2}}\right]$$

Extensions

Model

Multivariate co-integrated model:

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$$\mathbf{Y}(t) = \mathbf{L}_1(t) \in \mathbb{R}^m,$$

where

- $(\mathbf{L}_1(t))_{t\geq 0}$ is a m-dimensional Lévy process,

Model

Multivariate co-integrated model:

$$\begin{aligned} \mathbf{Y}(t) &= \mathbf{L}_1(t) \in \mathbb{R}^m, \\ \mathbf{X}(t) &= \mathbf{A}\mathbf{Y}(t) + \mathbf{Z}(t) \in \mathbb{R}^d, \end{aligned}$$

where

- $(\mathbf{L}_1(t))_{t\geq 0}$ is a m-dimensional Lévy process,
- $\mathbf{A} \in \mathbb{R}^{d\times m}$,

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where

- $(\mathbf{L}_1(t))_{t\geq 0}$ is a m-dimensional Lévy process,
- $\mathbf{A} \in \mathbb{R}^{d\times m}$,
- $(\mathbf{L}_2(t))_{t\geq 0}$ is a d-dimensional Lévy process,
- eigenvalues of $\mathbf{\Lambda} \in \mathbb{R}^{d\times d}$ have positive real parts,

Model

Multivariate co-integrated model:

$$\begin{aligned}
\mathbf{Y}(t) &= \mathbf{L}_1(t) + \zeta(t) \in \mathbb{R}^m, \\
\mathbf{X}(t) &= \mathbf{A}\mathbf{Y}(t) + \mathbf{Z}(t) \in \mathbb{R}^d, \\
\mathbf{Z}(t) &= \mathrm{e}^{-\mathbf{\Lambda} t}\mathbf{Z}(0) + \int_0^t \mathrm{e}^{-\mathbf{\Lambda}(t-s)}\,\mathrm{d}\mathbf{L}_2(s) \in \mathbb{R}^d
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$$\mathbf{Z}(t) = \text{multiv. cont.-time ARMA process}$$

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