Affine realizations for Lévy driven interest rate models with real-world forward rate dynamics

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Second NTH workshop on finance and insurance mathematics

July 2nd, 2011, Braunschweig

Introduction

- Markets of Zero Coupon Bonds $P(t,T)$.
- Forward rates under the *real-world measure* $\mathbb{P}$

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dX_t,$$

- driven by a *Lévy process* X with stochastic volatility.
- Existence of affine realizations

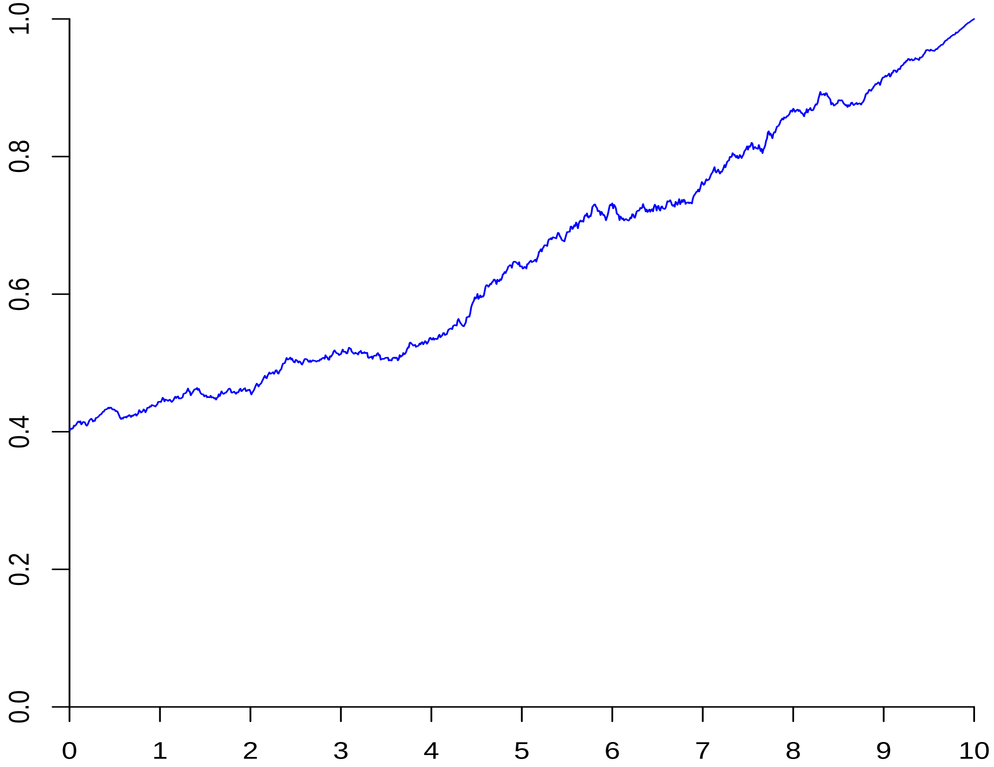
$$f(\omega;t,T) = \psi(t,T) + \sum_{i=1}^{d} Z_t^i(\omega)\phi_i(t,T).$$

Zero Coupon Bonds

- Zero Coupon Bond $P(t,T)$.
- Contract, which pays the holder one unit of cash at date T.

A typical price process

- Price process of a T-bond with maturity in 10 years:



The HJM methodology

- For every $T \geq 0$ the forward rates follow an Itô process

$$f(t,T) = f^*(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW_s, \quad t \in [0,T].$$

- Implied bond prices:

$$P(t,T) = \exp\Big(- \int_t^T f(t,s)ds \Big).$$

- Heath, Jarrow and Morton (HJM) 1992, see [11].

Absence of arbitrage

- Under an equivalent risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$ we have

$$f(t,T) = f^*(0,T) + \int_0^t \alpha_{\mathrm{HJM}}(s,T)ds + \int_0^t \sigma(s,T)dW_s, \quad t \in [0,T]$$

- where the drift is given by

$$\alpha_{\mathrm{HJM}}(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)ds.$$

- *Benchmark approach* from Platen, Heath (2006), see [13].

The benchmark approach

- We use the *growth optimal portfolio* S^* as numéraire.
- The real world measure $\mathbb{P}$ is the pricing measure.
- Existence of an equivalent risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$ is not required.
- Possible forward rate evolutions

$$f(t,T) = f^*(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW_s, \quad t \in [0,T].$$

- Existence of affine realizations with a driving Lévy process X.

The framework

- Consider a financial market

$$\begin{cases} dS_t^0 &= S_t^0 r_t dt \\ dS_t^1 &= S_t^1 (a_t dt + b_t dW_t). \end{cases}$$

- For a strategy $\delta = (\delta^0, \delta^1)$ we define the *portfolio*

$$S_t^\delta := \delta_t^0 S_t^0 + \delta_t^1 S_t^1, \quad t \geq 0.$$

- We shall only consider *self-financing* portfolios, i.e.

$$dS_t^\delta = \delta_t^0 dS_t^0 + \delta_t^1 dS_t^1.$$

Benchmarked portfolios as supermartingales

- There is a positive portfolio S^* with the following property:
- For each nonnegative portfolio S^δ the *benchmarked portfolio*

$$\hat{S}_t^\delta := \frac{S_t^\delta}{S_t^*}, \quad t \geq 0$$

is a nonnegative local martingale.

- Hence, $\hat{S}^\delta$ is a *supermartingale*.

The growth optimal portfolio

- The portfolio S^* satisfies the SDE

$$dS_t^* = S_t^* \big(r_t dt + \theta_t(\theta_t dt + dW_t) \big),$$

- where θ denotes the market price of risk

$$\theta_t = \frac{a_t - r_t}{b_t}, \quad t \geq 0.$$

- S^* is called the *growth optimal portfolio*, see Kelly 1956, [12].
- It maximizes the expected log-utility $\mathbb{E}[\ln(S_T^\delta)]$ for any $T > 0$.

Absence of arbitrage

- A nonnegative portfolio S^δ with $S_0^\delta = 0$ is an *arbitrage* if

$$\mathbb{P}(S_t^\delta > 0) > 0 \quad \text{for some } t > 0.$$

- *Note:* There does not exist arbitrage.

- Indeed, for a nonnegative portfolio S^δ with $S_0^\delta = 0$ we have

$$0 \leq \mathbb{E}[\hat{S}_t^\delta] \leq \mathbb{E}[\hat{S}_0^\delta] = 0. \quad (\hat{S}^\delta \text{ is a supermartingale})$$

- Since $\hat{S}_t^\delta = S_t^\delta / S_t^*$, $t \geq 0$ and S^* is positive, we obtain

$$\mathbb{P}(S_t^\delta > 0) = \mathbb{P}(\hat{S}_t^\delta > 0) = 0.$$

A classical form of arbitrage

- Arbitrage pricing theory:
  - Harrison and Kreps (1979), see [9];
  - Harrison and Pliska (1981), see [10];
  - Delbaen and Schachermayer (1994, 1998, 2006), see [4, 5, 6].
- Fundamental theorem of asset pricing:

$$\text{NFLVR} \quad \Leftrightarrow \quad \text{Equivalent risk-neutral measure } \mathbb{Q} \sim \mathbb{P} \text{ exists.}$$

- The previous no-arbitrage definition is weaker.

Real world pricing

- For an $\mathcal{F}_T$-measurable payoff profile H with $\frac{H}{S_T^*} \in L^1(\mathcal{F}_T)$ we define

$$\pi_t := S_t^* \mathbb{E}\left[\frac{H}{S_T^*} \,\middle|\, \mathcal{F}_t \right], \quad t \in [0,T].$$

- Then $\pi_T = H$ and the price process π is *fair*, that is

$$\hat{\pi}_t = \frac{\pi_t}{S_t^*}, \quad t \in [0,T]$$

is a martingale.

Economic motivation

- For a martingale M and a supermartingale X with $M_T = X_T$ we have

$$M_t = \mathbb{E}[M_T \,|\, \mathcal{F}_t] = \mathbb{E}[X_T \,|\, \mathcal{F}_t] \leq X_t, \quad t \in [0,T].$$

- Let S^δ be any nonnegative portfolio with $S_T^\delta = H$.

- Since $\hat{S}^\delta$ is a supermartingale, we have

$$\pi_t \leq S_t^\delta, \quad t \in [0,T].$$

- Hence, π_t is the economically correct price.

Risk-neutral pricing

- Suppose S^0 and S^1 are fair portfolios.
- Then S^1/S^0 is a martingale under $\mathbb{Q} \sim \mathbb{P}$ with

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \frac{\hat{S}_t^0}{\hat{S}_0^0}, \quad t \in [0,T].$$

- By Bayes' rule we obtain for a payoff profile H the formula

$$\pi_t = S_t^0 \mathbb{E}_{\mathbb{Q}}\left[\frac{H}{S_T^0} \,\bigg|\, \mathcal{F}_t \right], \quad t \in [0,T].$$

Zero Coupon Bonds

- A Zero Coupon Bond has the payoff profile $H \equiv 1$, hence

$$P(t,T) = S_t^* \mathbb{E}\left[\frac{1}{S_T^*} \,\middle|\, \mathcal{F}_t \right].$$

- Structure of the implied forward rates under the real world measure $\mathbb{P}$

$$f(t,T) = -\frac{\partial}{\partial T} \ln P(t,T).$$

The forward rates

- Under the real world measure $\mathbb{P}$ we have

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t,$$

- where the drift is of the form

$$\alpha(t,T) = -\theta_t \sigma(t,T) + \sigma(t,T) \int_t^T \sigma(t,s)ds.$$

- The HJM drift condition under $\mathbb{Q} \sim \mathbb{P}$ corresponds to $\theta \equiv 0$, that is

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)ds.$$

Driving Lévy process

- Now let X be a Lévy process.
- The growth optimal portfolio S^* satisfies the SDE

$$dS_t^* = S_t^* \bigg(r_t + \theta_t^2 + \int_{\mathbb{R}} \frac{(1 - e^{x\theta_t})^2}{e^{x\theta_t}} F(dx) \bigg) dt + \sqrt{c} S_t^* \theta_t dW_t$$
$$+ S_{t-}^* \int_{\mathbb{R}} \frac{1 - e^{x\theta_{t-}}}{e^{x\theta_{t-}}} (\mu^X(dt, dx) - F(dx)dt),$$

- where $c \geq 0$ is the Gaussian part and F the Lévy measure of X.

The forward rates

- Under the real world measure $\mathbb{P}$ we have

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dX_t,$$

- where the drift is of the form

$$\alpha(t,T) = \frac{d}{dT}\Psi\bigg(\theta_t - \int_t^T \sigma(t,s)ds\bigg) = -\sigma(t,T)\Psi'\bigg(\theta_t - \int_t^T \sigma(t,s)ds\bigg).$$

- The HJM drift condition under $\mathbb{Q} \sim \mathbb{P}$ corresponds to $\theta \equiv 0$, that is

$$\alpha(t,T) = \frac{d}{dT}\Psi\bigg(-\int_t^T \sigma(t,s)ds\bigg) = -\sigma(t,T)\Psi'\bigg(-\int_t^T \sigma(t,s)ds\bigg).$$

Musiela parametrization

- Musiela parametrization of forward rates

$$r_t(x) = f(t, t + x), \quad x \geq 0$$

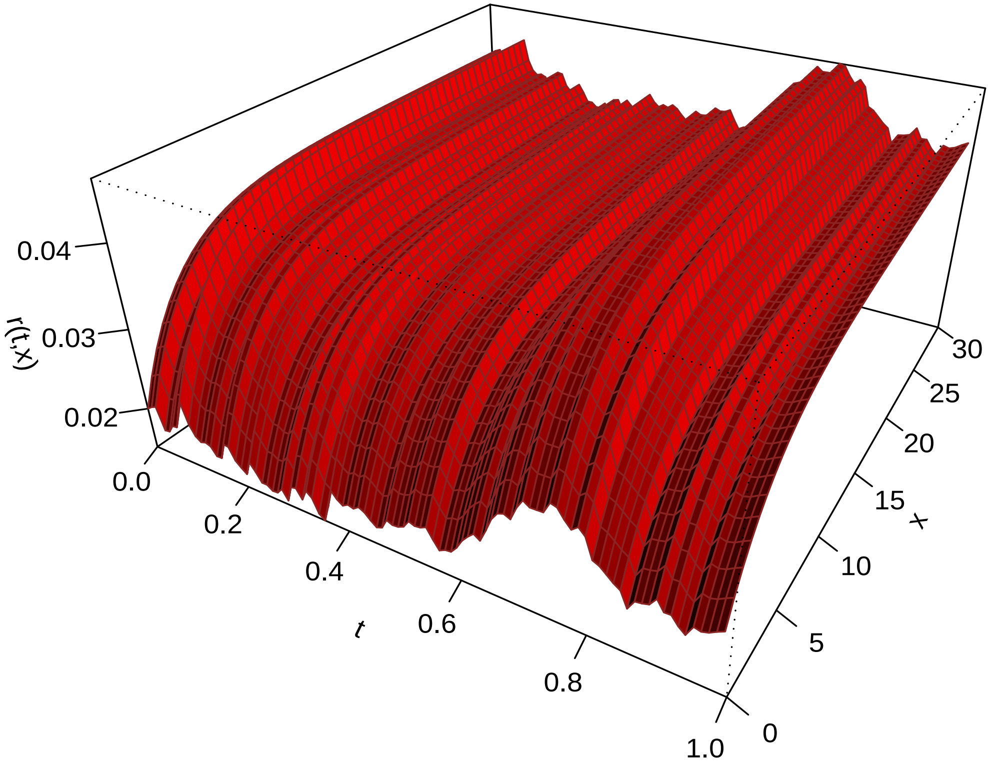
- Then $(r_t)_{t \geq 0}$ is a process in a state space H of forward curves.

- It satisfies the variation of constants formula

$$r_t = S_t f^*(0, \cdot) + \int_0^t S_{t-s} \alpha(s, s + \cdot) ds + \int_0^t S_{t-s} \sigma(s, s + \cdot) dX_s.$$

A typical forward curve evolution

- Forward curves over the period of 1 year:



The space of forward curves

- For $\beta > 0$ we define the separable Hilbert space

$$H_\beta := \{h : \mathbb{R}_+ \to \mathbb{R} \,|\, h \text{ is absolutely continuous with } \|h\|_\beta < \infty\},$$

- where the norm is given by

$$\|h\|_\beta := \left(|h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 e^{\beta x} dx \right)^{1/2}.$$

- The shift-semigroup $(S_t)_{t \geq 0}$ is C_0 on H_β with generator d/dx.

The HJMM equation

- Volatility depends on the forward curve r and an $\mathbb{R}^m$-valued process Z

$$\begin{cases} dr_t &= (\frac{d}{dx} r_t + \alpha(r_t, Z_t))dt + \sigma(r_{t-}, Z_{t-})dX_t \\ dZ_t &= b(r_t, Z_t)dt + c(r_{t-}, Z_{t-})dX_t \\ r_0 &= h_0 \\ Z_0 &= z_0. \end{cases}$$

- With $\theta : H_\beta \times \mathbb{R}^m \to \mathbb{R}$ being the market price of risk we have

$$\alpha(h,z) = \frac{d}{dx}\Psi\bigg(\theta(h,z) - \int_0^\bullet \sigma(h,z)(\eta)d\eta\bigg).$$

Existence of mild solutions

- The HJMM equation is an SPDE

$$\begin{cases} d\xi_t = (A\xi_t + \Theta(\xi_t))dt + \Sigma(\xi_{t-})dX_t \\ \xi_0 = h_0 \end{cases}$$

- on the state space $H_\beta \times \mathbb{R}^m$ with

$$A = \begin{pmatrix} d/dx & 0 \\ 0 & 0 \end{pmatrix}, \quad \Theta(h,z) = \begin{pmatrix} \alpha(h,z) \\ b(h,z) \end{pmatrix}, \quad \Sigma(h,z) = \begin{pmatrix} \sigma(h,z) \\ c(h,z) \end{pmatrix}.$$

- Existence and uniqueness of mild solutions under appropriate regularity.

Finite dimensional realizations

- The solution $(r_t)_{t \geq 0}$ has values in the infinite dimensional space H_β.
- We have a *finite dimensional realization* if

$$r_t = \phi(Y_t), \quad t \geq 0$$

with a mapping $\phi : \mathbb{R}^d \to H_\beta$ and an $\mathbb{R}^d$-valued process Y.

- There is a finite dimensional invariant manifold $\mathcal{M}$, that is

$$\mathbb{P}(r_t \in \mathcal{M}) = 1, \quad t \geq 0.$$

Affine realizations

- The parametrization $\phi : \mathbb{R}_+ \times \mathbb{R}^d \to H_\beta$ is of the form

$$\phi(t, y) = \psi(t) + \sum_{i=1}^{d} y_i h_i.$$

- Any model with a finite dimensional realization has an *affine realization*.
- Filipović, Teichmann (2003), see [7].
- Then $\mathcal{M} = (\mathcal{M}_t)_{t \geq 0}$ is a collection of affine spaces.

Foliations

- Let H be a Hilbert space and $V \subset H$ a finite dimensional subspace.
- A family $(\mathcal{M}_t)_{t \geq 0}$ of affine spaces is a *foliation* generated by V if

$$\mathcal{M}_t = \psi(t) + V, \quad t \geq 0$$

for some parametrization $\psi \in C^1(\mathbb{R}_+; H)$.

- The parametrization ψ is not unique, but we have

$$\psi_1(t) - \psi_2(t) \in V \quad \text{for all } t \geq 0.$$

Invariant foliations

- Now consider an H-valued SPDE

$$\begin{cases} dr_t = (Ar_t + \alpha(r_t))dt + \sigma(r_{t-})dX_t \\ r_0 = h. \end{cases}$$

- A foliation $(\mathcal{M}_t)_{t \geq 0}$ is *invariant* if for all $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{M}_{t_0}$ we have

$$\mathbb{P}(r_t \in \mathcal{M}_{t_0+t}) = 1, \quad t \geq 0$$

where $(r_t)_{t \geq 0}$ denotes the mild solution with $r_0 = h$.

The invariance result

- We define the tangent spaces

$$T\mathcal{M}_t := \psi'(t) + V, \quad t \geq 0.$$

- The foliation $(\mathcal{M}_t)_{t \geq 0}$ is invariant if and only if for all $t \geq 0$ we have

$$\mathcal{M}_t \subset \mathcal{D}(A),$$
$$\nu(h) := Ah + \alpha(h) \in T\mathcal{M}_t, \quad h \in \mathcal{M}_t$$
$$\sigma(h) \in V, \quad h \in \mathcal{M}_t.$$

- Note that the solution $(r_t)_{t \geq 0}$ cannot jump out of the foliation.

The state process

- Let ψ be a parametrization of $(\mathcal{M}_t)_{t\geq 0}$ and $\{\lambda_1, \ldots, \lambda_d\}$ a basis of V.
- Then, there exist unique $\mu, \gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\nu\bigg(\psi(t) + \sum_{i=1}^d y_i \lambda_i\bigg) = \psi'(t) + \sum_{i=1}^d \mu_i(t,y)\lambda_i,$$

$$\sigma\bigg(\psi(t) + \sum_{i=1}^d y_i \lambda_i\bigg) = \sum_{i=1}^d \gamma_i(t,y)\lambda_i.$$

- The $\mathbb{R}^d$-valued state process Y satisfies the SDE

$$dY_t = \mu(t, Y_t)dt + \gamma(t, Y_{t-})dX_t.$$

Definition of an affine realization

- Consider an H-valued SPDE

$$\begin{cases} dr_t = (Ar_t + \alpha(r_t))dt + \sigma(r_{t-})dX_t \\ r_0 = h. \end{cases}$$

- The SPDE has an *affine realization* generated by V if:
  - For each $h_0 \in \mathcal{D}(A)$ there exists an invariant foliation $(\mathcal{M}_t^{h_0})_{t \geq 0}$
  - generated by V
  - and with $h_0 \in \mathcal{M}_0^{h_0}$.

Consequences for affine realizations

- Let $V = \langle \lambda_1, \ldots, \lambda_d \rangle$ be a finite dimensional subspace.
- If the SPDE has an affine realization, then

$$\sigma(h) \in V, \quad h \in H.$$

- This implies that σ is of the form

$$\sigma(h) = \sum_{i=1}^{d} \Phi_i(h) \lambda_i, \quad h \in H.$$

Affine realizations for interest rate models

- Wiener driven models under an equivalent risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$:
  1. Björk et al. (2001, 2002, 2004), see [1, 2, 3];
  2. Filipović and Teichmann (2003, 2004), see [7, 8];
  3. Tappe (2010), see [15].
- Wiener driven models under the real world measure $\mathbb{P}$.
- Lévy driven models under an equivalent risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$.
- Lévy driven models under the real world measure $\mathbb{P}$.

Wiener driven models with real-world forward rates

- Consider the HJMM equation under $\mathbb{P}$ with stochastic volatility

$$\begin{cases} dr_t &= (\frac{d}{dx} r_t + \alpha(r_t, Z_t))dt + \sigma(r_t, Z_t)dW_t \\ dZ_t &= b(r_t, Z_t)dt + c(r_t, Z_t)dW_t \\ r_0 &= h_0 \\ Z_0 &= z_0, \end{cases}$$

- where the drift is given by

$$\alpha(h,z) = -\theta(h,z)\sigma(h,z) + \sigma(h,z)\int_0^{\bullet} \sigma(h,z)(\eta)d\eta.$$

The result

- The following statements are equivalent:

(1) There exists an affine realization;

(2) There exists an affine realization with $\theta \equiv 0$, i.e. with

$$\alpha(h, z) = \sigma(h, z) \int_0^{\bullet} \sigma(h, z)(\eta) d\eta.$$

- Hence, all known results for HJM models under $\mathbb{Q} \sim \mathbb{P}$ transfer.

Examples

- Suppose the volatility σ is of the form

$$\sigma(h,z) = \sum_{i=1}^{p} \Phi_i(h,z)\lambda_i,$$

- where the functions λ_i are *quasi-exponential*, i.e.

$$\dim\langle (d/dx)^n \lambda_i : n \in \mathbb{N}_0 \rangle < \infty.$$

- Then, there exists an affine realization for the HJMM equation.

Idea of the proof

- By our previous result, we require the condition

$$\sigma(h,z) \in V, \quad (h,z) \in H_\beta \times \mathbb{R}^m.$$

- Therefore, for all $t \geq 0$ we have

$$\sigma(h,z) \int_0^\bullet \sigma(h,z)(\eta) d\eta \in T\mathcal{M}_t, \quad (h,z) \in \mathcal{M}_t \times \mathbb{R}^m$$

if and only if

$$-\underbrace{\theta(h,z)}_{\in \mathbb{R}} \underbrace{\sigma(h,z)}_{\in V} + \sigma(h,z) \int_0^\bullet \sigma(h,z)(\eta) d\eta \in T\mathcal{M}_t, \quad (h,z) \in \mathcal{M}_t \times \mathbb{R}^m.$$

Risk-neutral Lévy driven models

- Consider the HJMM equation under $\mathbb{Q} \sim \mathbb{P}$ with stochastic volatility

$$\begin{cases} dr_t &= (\frac{d}{dx} r_t + \alpha(r_t, Z_t))dt + \sigma(r_{t-}, Z_{t-})dX_t \\ dZ_t &= b(r_t, Z_t)dt + c(r_{t-}, Z_{t-})dX_t \\ r_0 &= h_0 \\ Z_0 &= z_0, \end{cases}$$

- where the drift is given by the HJM drift condition

$$\alpha(h, z) = \frac{d}{dx} \Psi \bigg(- \int_0^\bullet \sigma(h, z)(\eta) d\eta \bigg).$$

Constant direction volatility

- Suppose the volatility is of the form

$$\sigma(h,z) = \Phi(h,z)\lambda.$$

- If $F(\mathbb{R}) \neq 0$, then the following statements are equivalent:

(1) There exists an affine realization;

(2) λ is quasi-exponential and for all $(h,z) \in H_\beta \times \mathbb{R}^m$ we have

$$D\Phi(h,z)((d/dx)^n \lambda, 0) = 0, \quad n \in \mathbb{N}$$
$$D\Phi(h,z)(0, e_k) = 0, \quad k = 1, \ldots, m.$$

- See Tappe (2011), [16].

Remarks

- In this case, the affine realization is generated by

$$V = \langle (d/dx)^n \lambda : n \in \mathbb{N}_0 \rangle.$$

- We have $\dim V < \infty$, because λ is quasi-exponential.

- The second condition means that for all $h \in H_\beta$ we have

$$\Phi \equiv \text{constant} \quad \text{on } h + V \times \mathbb{R}^m.$$

Risk-neutral Wiener driven models

- The previous result is due to the structure of the HJM drift term

$$\alpha(h,z) = \frac{d}{dx}\Psi\bigg(- \int_0^{\bullet} \sigma(h,z)(\eta)d\eta \bigg).$$

- Suppose $F(\mathbb{R}) = 0$. For constant direction volatilities

$$\sigma(h,z) = \Phi(h,z)\lambda$$

there exists an affine realization if:

  - λ is quasi-exponential (no condition on Φ) *or*
  - $\Lambda = \int_0^{\bullet} \lambda(\eta)d\eta$ satisfies a Riccati equation.

Lévy driven models with real-world forward rates

- Consider the HJMM equation under $\mathbb{P}$ with stochastic volatility

$$\begin{cases} dr_t &= (\frac{d}{dx} r_t + \alpha(r_t, Z_t))dt + \sigma(r_{t-}, Z_{t-})dX_t \\ dZ_t &= b(r_t, Z_t)dt + c(r_{t-}, Z_{t-})dX_t \\ r_0 &= h_0 \\ Z_0 &= z_0, \end{cases}$$

- where the drift is given by

$$\alpha(h,z) = \frac{d}{dx}\Psi\bigg(\theta(h,z) - \int_0^\bullet \sigma(h,z)(\eta)d\eta\bigg).$$

Constant direction volatility

- Conditions of the previous result and volatility

$$\sigma(h,z) = \Phi(h,z)\lambda.$$

- Assume $F(\mathbb{R}) \neq 0$ and that Ψ is *not* quasi-exponential. Equivalent:

(1) There exists an affine realization;

(2) For all $h \in H_\beta$ we have

$$\theta \equiv \text{constant} \quad \text{on } h + V \times \mathbb{R}^m.$$

- See Platen and Tappe (2011), [14].

Remarks

- X Wiener process, then $\Psi(x) = \frac{x^2}{2}$ is quasi-exponential.
- X Poisson process, then $\Psi(x) = e^x - 1$ is quasi-exponential.
- For $\sigma \equiv 1$ and $\theta : H_\beta \times \mathbb{R}^m \to \mathbb{R}$ arbitrary we have

$$\alpha(h,z)(x) = -e^{\theta(h,z)} e^{-x}, \quad x \geq 0.$$

- Hence, there is an affine realization generated by $V = \langle 1, e^{-x} \rangle$.

Conclusion

- Affine realizations for interest rate models with stochastic volatility.
- Driving Wiener process under the real-world measure $\mathbb{P}$:

  The same results as in the risk-neutral world.
- Driving Lévy process under a risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$:

  Restrictions on the volatility σ.
- Driving Lévy process under the real-world measure $\mathbb{P}$:

  Further restrictions on the market price of risk θ.

References

[1] Björk, T., Svensson, L. (2001): On the existence of finite dimensional realizations for nonlinear forward rate models. *Mathematical Finance* **11**(2), 205–243.

[2] Björk, T., Landén, C. (2002): On the construction of finite dimensional realizations for nonlinear forward rate models. *Finance and Stochastics* **6**(3), 303–331.

[3] Björk, T., Landén, C., Svensson, L. (2004): Finite-dimensional Markovian realizations for stochastic volatility forward-rate models. Stochastic analysis with applications to mathematical finance. *Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences* **460**(2041), 53-83.

[4] Delbaen, F., Schachermayer, W. (1994): A general version of the

fundamental theorem of asset pricing. *Mathematische Annalen* **300**, 463–520.

[5] Delbaen, F., Schachermayer, W. (1998): The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen* **312**, 215–250.

[6] Delbaen, F., Schachermayer, W. (2006): *The mathematics of arbitrage.* Springer, Berlin.

[7] Filipović, D., Teichmann, J. (2003): Existence of invariant manifolds for stochastic equations in infinite dimension. *Journal of Functional Analysis* **197**(2), 398–432.

[8] Filipović, D., Teichmann, J. (2004): On the geometry of the term structure of interest rates. *Proceedings of The Royal Society of London.*

Series A. Mathematical, Physical and Engineering Sciences **460**(2041), 129–167.

[9] Harrison, J. M., Kreps, D. M. (1979): Martingale and arbitrage in multiperiod securities markets. *Journal of Economic Theory* **20**, 381–408.

[10] Harrison, J. M., Pliska, S. R. (1981): Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and Their Applications* **11**(3), 215–260.

[11] Heath, D., Jarrow, R., Morton, A. (1992): Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica* **60**(1), 77–105.

[12] Kelly, J. R. (1956): A new interpretation of information rate. *Bell Syst. Techn. J.* **35**, 917–926.

[13] Platen, E., Heath, D. (2006): *A Benchmark Approach to Quantitative Finance.* Springer, Berlin.

[14] Platen, E., Tappe, S. (2011): *Affine realizations for Lévy driven interest rate models with real-world forward rate dynamics.* Preprint, submitted.

[15] Tappe, S. (2010): An alternative approach on the existence of affine realizations for HJM term structure models. *Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences* **466**(2122), 3033–3060.

[16] Tappe, S. (2011): *Existence of affine realizations for Lévy term structure models.* Preprint, submitted.