

Risk Measures for Solvency Regulation and Asset Liability Management

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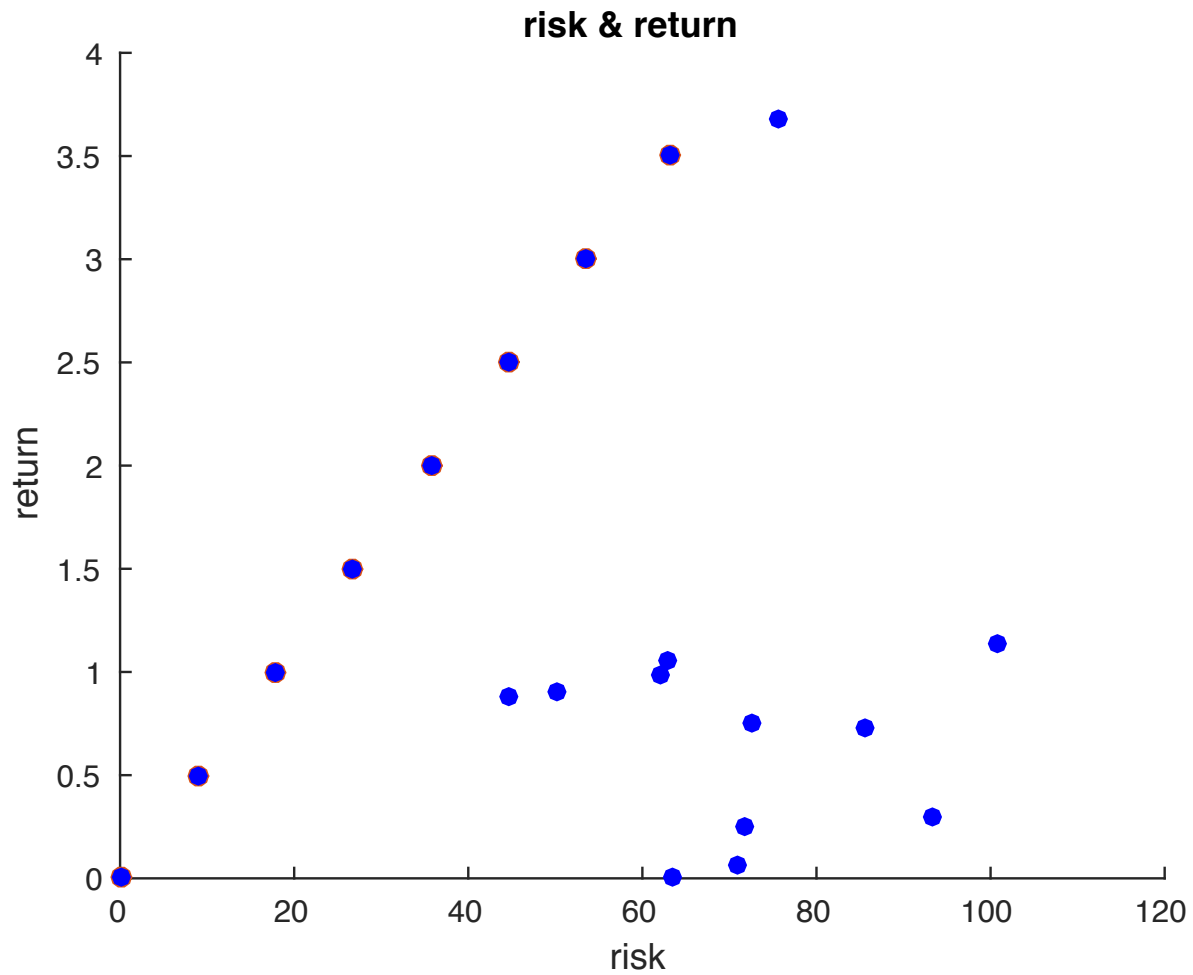
Motivation

- A key task of insurance companies consists in managing and balancing **risk and return**
- Important quantities are, for example, the
 - **Market-Consistent Embedded Value** (MCEV)
as a quantitative measure of the **value of the current business**
and
 - **Value at Risk** (V@R) or **Average Value at Risk** (AV@R)
as quantitative measures of the **downside risk**.

Motivation (2)

- The optimization of risk and return requires adequate strategies for the **management of the assets and the liabilities** of the balance sheet.
- A comprehensive analysis is quite sophisticated:
 - Quantities like **MCEV** (VIF, FS, ReC), **SCR**,... need to be computed.
 - The impact of **dynamic management rules** must be characterized.
 - This requires **adequate ALM-models**.

Efficient Frontier



Outline

- (i) **Short review of Solvency II – Pillar I:**
Quantitative Requirements
- (ii) **Alternative risk measures:**
AV@R, UBSR
- (iii) **Statistical risk measurement:**
Risk regression and robustness
- (iv) **Future research:**
Systemic risk and group risk

Solvency II

Solvency II

- **Main aim:** more appropriate and better risk measurement
- **Methodology:**
 - Standard formula or internal model
 - Market or market-consistent values instead of accounting measures
 - Three pillar approach

Three Pillars

- **I. Quantitative Requirements**
- **II. Supervisory Review**
- **III. Disclosure**

Three Pillars (2)

- **I. Quantitative Requirements**

- Market-consistent valuation of assets and liabilities
- Computation of the solvency capital requirement on the basis of the standard formula or an internal model

- II. Supervisory Review

- III. Disclosure

The Solvency Balance Sheet

- **The role of capital**
 - Buffer for potential losses
 - that protects policy holders (and other counterparties)
- **Solvency balance sheet**
 - Market-consistent valuation of all assets and liabilities
- **SCR = Solvency Capital Requirement**
 - Key goal: Limit one-year probability of ruin to at most 0.5%.

The Solvency Balance Sheet (2)

Assets

- Market value of all assets

Liabilities

- Economic capital
 - i.e. SCR + Free Surplus
- Non-hedgeable liabilities
 - Best Estimate
 - Risk Margin
- Hedgeable liabilities
 - Market-Consistent Value

SCR

The **SCR** corresponds to the economic capital a (re)insurance undertaking needs to hold in order to limit the probability of ruin to 0.5%, i.e. ruin would occur once every 200 years. . .

The **SCR** is calculated using Value-at-Risk techniques, either in accordance with the standard formula, or using an internal model: all potential losses, including adverse revaluation of assets and liabilities, over the next 12 months are to be assessed. The **SCR** reflects the true risk profile of the undertaking, taking account of all quantifiable risks, as well as the net impact of risk mitigation techniques.

(Proposal for a Directive of the European Parliament and of the Council on the taking-up and pursuit of the business of Insurance and Reinsurance - Solvency II, COMMISSION OF THE EUROPEAN COMMUNITIES, Brussels, 10.7.2007)

SCR in a Simplified Internal Model

- **Time:** $t = 0, 1$
- **Value of assets:** $A_t, t = 0, 1$
- **Value of liabilities:** $L_t, t = 0, 1$
- **Capital (NAV):** $E_t = A_t - L_t, t = 0, 1$
- **Level:** $\alpha = 0.05$

$$P(E_1 \leq 0) \leq \alpha$$

$$\Leftrightarrow V@R_\alpha(E_1) \leq 0 \Leftrightarrow V@R_\alpha(E_1 - E_0) \leq E_0 \Leftrightarrow V@R_\alpha(\Delta A_1 - \Delta L_1) \leq E_0,$$

with $\Delta A_1 = A_1 - A_0$, $\Delta L_1 = L_1 - L_0$.

Cash flows (premiums, taxes, etc.) are assumed to be implicitly included.

Simplified NAV instead of MCEV computation, thus VIF neglected.

SCR in a Simplified Internal Model (2)

As a consequence the **SCR** is defined as follows:

$$\mathbf{SCR} = \mathbf{V@R}_\alpha(\Delta\mathbf{A}_1 - \Delta\mathbf{L}_1) = \mathbf{V@R}_\alpha(\Delta\mathbf{NAV})$$

The **solvency condition** can be reformulated:

$$\mathbf{SCR} \leq \mathbf{E}_0.$$

Alternative Risk Measures

Monetary Risk Measures

- Model for one time period as in Solvency II: $t = 0, 1$
- \mathcal{X} is space of positions at time 1 modeled by random variables (P&L)

Risk measures

$$\rho : \mathcal{X} \rightarrow \mathbb{R}$$

- **Monotonicity:** If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- **Cash invariance:** If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.

A risk measure is statistics that summarizes certain properties of random future balance sheets.

Risk measures like V@R focus on the downside risk.

Capital requirements

- A position $X \in \mathcal{X}$ is **acceptable**, if $\rho(X) \leq 0$.
The collection \mathcal{A} of all acceptable positions is the *acceptance set*.
- ρ is a **capital requirement**, i.e.

$$\rho(X) = \inf \{m \in \mathbb{R} : X + m \in \mathcal{A}\}.$$

Example

$$\text{V@R}_\lambda(X) = \inf \{m \in \mathbb{R} : P[m + X < 0] \leq \lambda\}$$

“Smallest monetary amount that needs to be added to a position such that the probability of a loss becomes smaller than λ .”

Diversification

Semiconvexity:

$$\rho(\alpha X + (1 - \alpha)Y) \leq \max(\rho(X), \rho(Y)) \quad (\alpha \in [0, 1]).$$

\implies

Convexity (Föllmer & Schied, 2002):

$$\rho(\alpha X + (1 - \alpha)Y) \leq \alpha\rho(X) + (1 - \alpha)\rho(Y) \quad (\alpha \in [0, 1]).$$

Average Value at Risk

$$AV@R_\lambda(X) = \frac{1}{\lambda} \int_0^\lambda V@R_\gamma(X) d\gamma$$

Properties

- coherent (i.e. convex and positively homogeneous)
- sensitive to large losses
- basis of Swiss Solvency test
- common alternative to V@R in practice
- distribution-based and continuous from above
- building block of large class of risk measures

Utility-based Shortfall Risk (UBSR)

$\ell : \mathbb{R} \rightarrow \mathbb{R}$ convex loss function, z interior point of the range of ℓ .

The **acceptance set** is defined as

$$\mathcal{A} = \{X \in L^\infty : E_P[\ell(-X)] \leq z\}$$

\mathcal{A} induces a **convex** risk measure ρ :

$$\rho(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}$$

Simple formula

Shortfall risk $\rho(X)$ is given by the **unique root** s_* of the function

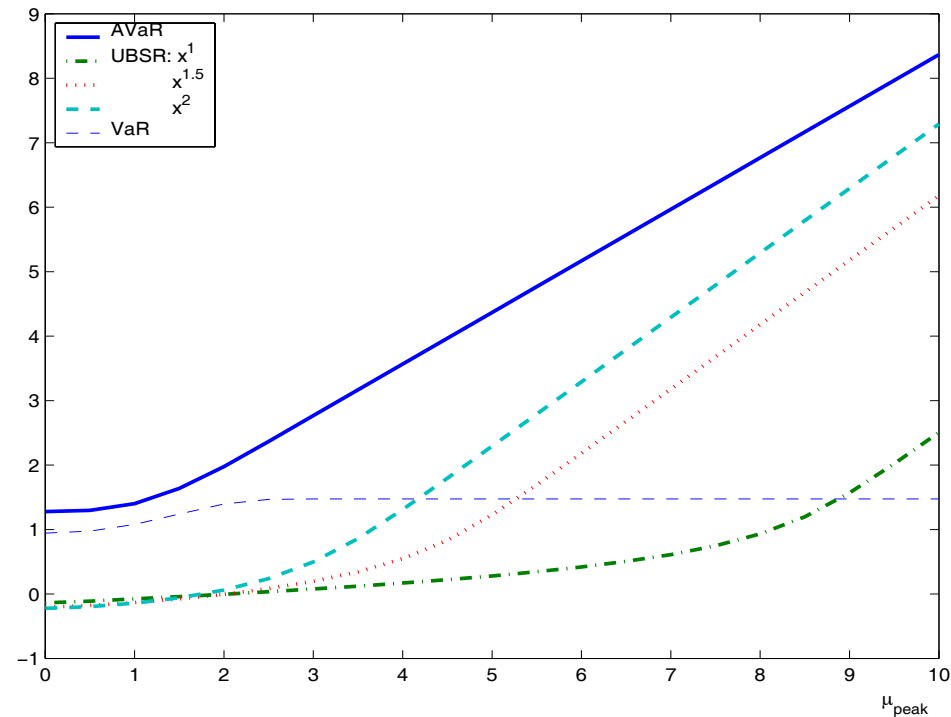
$$f(s) := E[\ell(-X - s)] - z.$$

Utility-based Shortfall Risk (UBSR)

Properties

- convex
- sensitive to large losses
- distribution-based and continuous from above
- easy to estimate and implement (see below)
- elicitable (see below)

Sensitivity to the Downside Risk



$V@R_{0.05}$, $AV@R_{0.05}$ and UBSR with $p \in \{1, \frac{3}{2}, 2\}$ and $z = 0.3$ of a mixture of a Student t (weight 0.96) and a Gaussian with mean μ (weight 0.04) as a function of μ .

Value at Risk in the Media

“[David Einhorn](#), who founded Greenlight Capital, a prominent hedge fund, wrote not long ago that VaR was

'relatively useless as a risk-management tool and potentially catastrophic when its use creates a false sense of security among senior managers and watchdogs. This is like an air bag that works all the time, except when you have a car accident.' ”

“[Nicholas Taleb](#), the best-selling author of 'The Black Swan,' has crusaded against VaR for more than a decade. He calls it, flatly, '*a fraud.*' ”

(“Risk Mismanagement”, New York Times, 2. Januar 2009)

Application to SCR

SCR in a Simplified Internal Model

- **Time:** $t = 0, 1$
- **Value of assets:** $A_t, t = 0, 1$
- **Value of liabilities:** $L_t, t = 0, 1$
- **Capital (NAV):** $E_t = A_t - L_t, t = 0, 1$

$$P(E_1 \leq 0) \leq \alpha$$

$$\Leftrightarrow E_1 \in \mathcal{A}_{V@R_\alpha}$$

$$\Leftrightarrow SCR := V@R_\alpha(\Delta A_1 - \Delta L_1) \leq E_0,$$

with $\Delta A_1 = A_1 - A_0, \Delta L_1 = L_1 - L_0$.

Generalized SCR

- **Time:** $t = 0, 1$
- **Value of assets:** $A_t, t = 0, 1$
- **Value of liabilities:** $L_t, t = 0, 1$
- **Capital (NAV):** $E_t = A_t - L_t, t = 0, 1$

$$\rho(E_1) \leq 0$$

$$\Leftrightarrow E_1 \in \mathcal{A}_\rho$$

$$\Leftrightarrow SCR := \rho(\Delta A_1 - \Delta L_1) \leq E_0,$$

with $\Delta A_1 = A_1 - A_0, \Delta L_1 = L_1 - L_0$.

Generalized SCR (2)

- **Requirement:** E_1 acceptable with respect to risk measure ρ
- **Solvency capital requirement**
is equivalent to

$$\rho(\Delta NAV) = \rho(\Delta A_1 - \Delta L_1) \leq E_0$$

(Proof: as previous computation)

Example: SCR and AV@R

Using **AV@R** (which is used in the SST) generates the SCR:

$$\mathbf{AV@R}_\lambda(\Delta NAV) = \mathbf{AV@R}_\lambda(\Delta A_1 - \Delta L_1)$$

In this case, the solvency requirement does not focus on the probability of insolvency, but on **the average loss in the case of insolvency**.

- From a macroeconomic point of view, this seems to be a reasonable approach.
- Alternatively, **UBSR** could be used instead of V@R or AV@R.

Statistics of Risk: Backtesting and Robustness

Motivation

Risk measures should possess **adequate statistical properties** for their estimation and a comparison to data:

- **Elicitability**
 - Good properties in the context of backtesting? (Gneiting, 2011)
 - Generalized quantile regression methods (Koenker, 2005)
- **Robustness**
 - Robust computation in case of slightly incorrect models

Elicitability

A **scoring function** $S : \mathbb{R}^2 \rightarrow [0, \infty)$ satisfies:

- $S(x, y) \geq 0$, and $S(x, y) = 0 \Leftrightarrow x = y$
- $x \mapsto S(x, y)$ increasing for $x > y$ and decreasing for $x < y$
- $x \mapsto S(x, y)$ continuous

Definition 1 A statistical functional $T : \mathcal{M} \rightarrow \mathbb{R}$ is **elicitable** on \mathcal{M} , if there exists a scoring function S such that for each $\mu \in \mathcal{M}$:

$$\int S(x, y) \mu(dy) < \infty,$$

$$\mathbf{T}(\mu) = \operatorname{argmin}_{\mathbf{x} \in \mathbb{R}} \int \mathbf{S}(\mathbf{x}, \mathbf{y}) \mu(d\mathbf{y})$$

Characterization

- An application of results of W. (2006) implies a complete characterization of all **elicitable risk measures** under weak technical conditions on the scoring function S .
- The details are worked out in Bellini & Bignozzi (2015).
- Delbaen, Bellini, Bignozzi & Ziegel (2015) investigate the special case of convex risk measures which does not require additional topological assumptions.

Characterization (2)

Theorem 1 *Let ρ be a risk measure that is elicitable for a regular scoring function.*

Then the following statements hold:

(i) ρ is *convex*, if and only if ρ is UBSR.

(ii) ρ is *coherent*, if and only if ρ is an *expectile*.

Conversely, UBSR is always elicitable for a regular scoring function.

Remark

AV@R is not elicitable. (W., 2006; Gneiting, 2011)

Expectile

Special case of coherent UBSR with piecewise linear loss function

$$l(x) = z + \alpha x^+ - \beta x^-,$$

$x \in \mathbb{R}$, $\alpha \geq \beta > 0$ with level $z \in \mathbb{R}$.

Acceptability

A position $X \in L^\infty$ is **acceptable**, if and only if

$$\beta E(X^+) - \alpha E(X^-) \geq 0,$$

i.e. a difference between weighted expected gains and weighted expected losses is larger than 0.

Remark

AV@R and expectiles are not surplus invariant.

Robustness

Robustness

- Convex risk measures have recently been criticized, since they are **not Hampel-robust**, see e.g. Cont, Deguest & Scandolo (2010) and Kou, Peng & Heyde (2013).
- This notion of robustness is, however, related to the **weak topology** and formalized in terms of a metrization like the **Lévy or Prohorov metric which are tail-insensitive**.
- By Hampel's theorem, Hampel-robustness implies continuity with respect to the weak topology and, thus, **insensitivity to the tails**.

Robustness

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Question:

Should we measure the downside risk of financial positions with tail-insensitive, but Hampel-robust functionals?

Robustness (2)

- **Robustness** depends on the **metric** that is used on the space of probability measures.
- Risk measures are not either robust or not robust, but **more or less robust**.
- \exists **tradeoff** between tail sensitivity and robustness.
- These issues were studied by Krätschmer, Schied & Zähle (2014).
- Here: **application in the context of elicitable risk measures**.

Distribution-based Risk Measures

A risk measure $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is **distribution-based**, if

$$\rho(X) = \rho(Y),$$

whenever $\mathcal{L}(X) = \mathcal{L}(Y)$.

Risk measures on the level of distributions

In this case, the risk measure defines a map

$$\mathcal{R}_\rho(\mu) := \rho(X) \text{ with } \mathcal{L}(X) = \mu$$

on Borel probability measures.

Qualitative Robustness

- Let $\Omega = \mathbb{R}^{\mathbb{N}}$, $X_i(\omega) = \omega(i)$, and $\mathcal{F} = \sigma(X_i : i \in \mathbb{N})$.
- For any Borel probability measure μ on \mathbb{R} denote

$$P_\mu := \mu^{\otimes \mathbb{N}}.$$

- $\mathcal{N} \subseteq \mathcal{M}_1$ with metric d_A
- d_B metric on \mathcal{M}_1

\mathcal{R}_ρ robust on \mathcal{N} with respect to d_A, d_B , if

$$\forall \mu \in \mathcal{N}, \varepsilon > 0 \exists \delta > 0, n_0 \in \mathbb{N} \forall n \geq n_0 :$$

$$\nu \in \mathcal{N}, d_A(\mu, \nu) \leq \delta \Rightarrow d_B(P_\mu \circ \hat{\rho}_n^{-1}, P_\nu \circ \hat{\rho}_n^{-1}) \leq \varepsilon$$

where $\hat{\rho}_n := \mathcal{R}_\rho\left(\frac{1}{n} \sum_{k=1}^n \delta_{X_k}\right)$

ψ -Robustness

A set $\mathcal{N} \subset \mathcal{M}_1$ is called uniformly ψ -integrating, if

$$\lim_{M \rightarrow \infty} \sup_{\nu \in \mathcal{N}} \int_{\{\psi \geq M\}} \psi d\nu = 0.$$

Definition 2

A risk functional \mathcal{R}_ρ is called *ψ -robust on $\mathcal{M} \subset \mathcal{M}_1$* ,

if it is robust with respect to d_ψ and d_{Proh} on every uniformly ψ -integrating set $\mathcal{N} \subset \mathcal{M}$.

ψ -Robustness (2)

The Prohorov- ψ -metric is

$$d_{\psi}(\mu, \nu) = d_{\text{Proh}}(\mu, \nu) + \left| \int \psi d\mu - \int \psi d\nu \right|, \quad \mu, \nu \in \mathcal{M}_1^{\psi},$$

where the Prohorov-metric is defined by

$$d_{\text{Proh}}(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(A) \leq \nu(A^{\varepsilon}) + \varepsilon \text{ for all } A \in \mathcal{B}(\mathbb{R})\}$$

with $A^{\varepsilon} = \{x \in \mathbb{R} : \inf_{a \in A} |x - a| \leq \varepsilon\}$, $\varepsilon > 0$.

While the weak topology is induced by the Prohorov-metric and is insensitive with respect to extreme events, the ψ -weak topology is sensitive with respect to such events.

ψ -Robustness (3)

Theorem 2 Let $\rho : L^\infty \rightarrow \mathbb{R}$ be a distribution-based risk measure and $\bar{\rho} : L^1 \rightarrow \mathbb{R} \cup \{+\infty\}$ its unique extension. Suppose that Ψ is a *finite Young-function* that satisfies the Δ_2 -condition. Define $\psi(x) = 1 + \Psi(|x|)$, $x \in \mathbb{R}$.

Then the following conditions are equivalent:

- $\bar{\rho}$ is *finite on H^Ψ* .
- The mapping $\mathcal{R}_\rho : \mathcal{M}_{1,c} \rightarrow \mathbb{R}$ is *continuous with respect to the ψ -weak topology*.
- $\mathcal{R}_\rho : \mathcal{M}_{1,c} \rightarrow \mathbb{R}$ is *ψ -robust*.

Index of Qualitative Robustness

L^p -spaces are particularly important for applications. Let $\Psi_p(x) = |x|^p/p$ with $0 < p < \infty$, $x \in \mathbb{R}$, then the Δ_2 -condition is always satisfied.

Definition 3 Let $\rho : L^\infty \rightarrow \mathbb{R}$ be a distribution-based risk measure. The associated **index of qualitative robustness** is defined as

$$\text{iqr}(\rho) = \frac{1}{\inf \{p \in (0, \infty) : \mathcal{R}_\rho \text{ is } \psi_p\text{-robust on } \mathcal{M}_{1,c}\}} \in [0, \infty]$$

Convex risk measures have an index of qualitative robustness of **at most 1**.

Theorem 2 implies the following formula:

$$\text{iqr}(\rho) = \frac{1}{\inf \{p \in [1, \infty) : \bar{\rho} \text{ is finite on } L^p\}} \in [0, 1] \quad (1)$$

Elicitability and Robustness

Convex elicitable risk measures coincide with the class of utility-based shortfall risk measures:

$$\rho(X) = \inf \{m \in \mathbb{R} : E(\ell(-X - m)) \leq z\}$$

with convex, increasing, non-constant function $\ell : \mathbb{R} \rightarrow \mathbb{R}$ and z in the interior of the range of ℓ .

- Set $\Psi(x) = \ell(x) - \ell(0)$, $x \geq 0$, $\psi(x) = \Psi(|x|) + 1$, $x \in \mathbb{R}$. Assume that the function Ψ satisfies the Δ_2 -condition.
- Elementary bounds show that \mathcal{R}_ρ is continuous with respect to the ψ -weak topology and thus ψ -robust.

Elicitability and Robustness (2)

Expectiles

Special case of UBSR with piecewise linear loss function

$$\ell(x) = z + \alpha x^+ - \beta x^-, \quad x \in \mathbb{R}, \quad \alpha \geq \beta > 0 \text{ with level } z \in \mathbb{R}.$$

- **Finite** distribution-based risk measures on L^1 .
- **Index of qualitative robustness is 1.**
- Expectiles are as robust as Average Value at Risk.

Elicitability and Robustness (3)

Monomials

For $\ell(x) = x^p \mathbf{1}_{\{x \geq 0\}}$ with Level $z > 0$ the associated UBSR has the following properties:

- ρ is finite on L^p with an index of qualitative robustness of $1/p$.
- The corresponding functional \mathcal{R}_ρ is ψ_p -robust.

Entropic risk measure

- The entropic risk measure is never finite on L^p , $1 \leq p < \infty$.
- Equation (1) shows that its index of qualitative robustness is 0.

Future Research

Systemic Risk and Group Risk

- Key objectives of the **regulation of financial systems** and the **management of insurance groups** are **upper bounds on overall risk** in the system or group.
- **Capital levels for individual entities cannot be chosen independently** of each other, but jointly influence overall risk.
- A similar effect occurs when a company has subsidiaries in countries with different currencies.
- These issues can be captured by **set-valued risk measures**.

Conclusion

Conclusion

- **V@R has significant deficiencies.** These deficiencies are inherited by the SCR-regulation under Solvency II.
- **AV@R** and **expectiles** are reasonable additional functionals that characterize the downside risk and can easily be implemented.

Recommendation

The analysis of the downside risk should involve a joint analysis of V@R, AV@R and expectiles.

Future Research

Systems of entities require set-valued risk measures for a proper characterization of their risk.

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