Statistical inference for risk measures

Henryk Zähle
Saarland University

DAA-Workshop für junge Mathematiker
Schloss Reisensburg
September 4–6, 2014

Based on joint work with

Eric Beutner (Maastricht), Volker Krätschmer (Essen), Alexandra Lauer (Saarbrücken), Alexander Schied (Mannheim), and Wei Biao Wu (Chicago)

Contents

1. Introduction

What is a risk measure?

Let X be a risky position, e.g.

1. the total claim of an insurance collective in the next period,
2. the shortfall of the equity of an insurer at a reporting date (= "cash value" of liabilities – "cash value" of assets).

A monetary risk measure is a "suitable" map $\rho : \mathcal{X} \longrightarrow \mathbb{R}$ which assigns to the position $X \in \mathcal{X}$ the minimal amount of capital $\rho(X)$ which, if added to the position and invested in a risk-free manner, makes the position "acceptable".

A position $Y \in \mathcal{X}$ is "acceptable" with respect to ρ if $\rho(Y) \leq 0$.

$\rho(X - \rho(X)) = \rho(X) - \rho(X) = 0$ $\quad$ ($\rightarrow$ axiom (2)).

1. Introduction

What is a risk measure?

Let X be a risky position, e.g.

1. the total claim of an insurance collective in the next period,
2. the shortfall of the equity of an insurer at a reporting date (= "cash value" of liabilities − "cash value" of assets).

A monetary risk measure is a "suitable" map $\rho : \mathcal{X} \longrightarrow \mathbb{R}$ which assigns to the position $X \in \mathcal{X}$ the minimal amount of capital $\rho(X)$ which, if added to the position and invested in a risk-free manner, makes the position "acceptable". E.g.

1. $\rho(X)$ = premium for the whole collective
2. $\rho(X)$ = solvency capital requirement (SCR) of insurer

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be atomless, and $\mathcal{X} \subset L^0(\Omega, \mathcal{F}, \mathbb{P})$ be a vector space containing the constants. Let $\rho : \mathcal{X} \longrightarrow \mathbb{R}$ be a map, and consider the following conditions:

(1) monotonicity: $\rho(X_1) \leq \rho(X_2)$ for all $X_1, X_2 \in \mathcal{X}$ with $X_1 \leq X_2$.

(2) cash additivity: $\rho(X + m) = \rho(X) + m$ for all $X \in \mathcal{X}$ and $m \in \mathbb{R}$.

(3) subadditivity: $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$ for all $X_1, X_2 \in \mathcal{X}$.

(4) positive homogeneity: $\rho(\lambda X) = \lambda \, \rho(X)$ for all $X \in \mathcal{X}$ and $\lambda \geq 0$.

ρ is a monetary risk measure if (1)–(2) hold.

ρ is a coherent risk measure if (1)–(4) hold.

ρ is law-invariant if $\rho(X_1) = \rho(X_2)$ whenever $\mathbb{P}_{X_1} = \mathbb{P}_{X_2}$.

Example 1 The expectation

$$\mathbb{E}[X] := \int X \, d\mathbb{P} = \int x \; dF_X(x)$$

is a law-invariant coherent risk measure on $\mathcal{X} = L^1(\Omega, \mathcal{F}, \mathbb{P})$.
It has the representation

$$\mathbb{E}[X] = -\int_{-\infty}^{0} F_X(x)\, dx + \int_{0}^{\infty} \big(1 - F_X(x)\big)\, dx.$$

Example 2 The Value at Risk at level $\alpha \in (0, 1)$

$$\mathrm{V@R}_\alpha(X) := F_X^{\leftarrow}(1-\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq 1-\alpha\}$$

is a law-invariant and positively homogeneous monetary risk measure on $\mathcal{X} = L^0(\Omega, \mathcal{F}, \mathbb{P})$. But it is not subadditive, hence not coherent.

Example 3 The Average Value at Risk at level $\alpha \in (0,1)$

$$\mathrm{AV@R}_\alpha(X) := \frac{1}{\alpha}\int_0^\alpha \mathrm{V@R}_s(X)\,ds$$

is a law-invariant coherent risk measure on $\mathcal{X} = L^1(\Omega, \mathcal{F}, \mathbb{P})$.
If F_X is continuous at $\mathrm{V@R}_\alpha(X)$, then

$$\mathrm{AV@R}_\alpha(X) = \mathbb{E}[X | X \geq \mathrm{V@R}_\alpha(X)].$$

Example 4 Let g be a convex distortion function, i.e. a convex nondecreasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$. The distortion risk measure associated with g

$$\rho_g(X) \;:=\; -\int_{-\infty}^{0} g(F_X(x))\,dx + \int_{0}^{\infty} \big(1 - g(F_X(x))\big)\,dx$$

is a law-invariant coherent risk measure on $\mathcal{X} = \mathcal{X}_g := \{\cdots\}$.
For right-continuous g, we have the representations

$$\rho_g(X) \;=\; \int_0^1 \mathrm{V@R}_s(X)\,dg(s) \;=\; \int_{-\infty}^{\infty} x\;d(g \circ F_X)(x).$$

If specifically $g(t) = \max\{(t-(1-\alpha))/\alpha; 0\}$, then $\rho_g = \mathrm{AV@R}_\alpha$.

Example 4 Let g be a convex distortion function, i.e. a convex nondecreasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$. The distortion risk measure associated with g

$$\rho_g(X) := -\int_{-\infty}^{0} g(F_X(x))\,dx + \int_0^{\infty} \big(1 - g(F_X(x))\big)\,dx$$

is a law-invariant coherent risk measure on $\mathcal{X} = \mathcal{X}_g := \{\cdots\}$.
For right-continuous g, we have the representations

$$\rho_g(X) = \int_0^1 \mathrm{V@R}_s(X)\,dg(s) = \int_{-\infty}^{\infty} x\; d(g \circ F_X)(x).$$

Distortion risk measures associated with convex distortion functions are the building blocks of rather general law-invariant coherent risk measures . . .

Theorem Let ρ be a law-invariant coherent risk measure on $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $p \in [1, \infty]$. Then there is some set $\mathcal{G}_\rho$ of continuous convex distortion functions such that

$$\rho(X) = \sup_{g \in \mathcal{G}_\rho} \rho_g(X) \qquad \text{for all } X \in \mathcal{X}$$

"Kusuoka representation".

Kusuoka (2001)
Föllmer/Schied (2004)
Krätschmer/H. Z. (2011)
Belomestny/Krätschmer (2012)

For law-invariant ρ and $\mathcal{M}(\mathcal{X}) := \{\mathbb{P}_X : X \in \mathcal{X}\}$, we may define a map

$$\mathcal{R}_\rho : \mathcal{M}(\mathcal{X}) \longrightarrow \mathbb{R} \qquad \text{by} \qquad \mathcal{R}_\rho(\mathfrak{m}) := \rho(X_\mathfrak{m}),$$

where $X_\mathfrak{m} \in \mathcal{X}$ has law $\mathfrak{m}$. We call $\mathcal{R}_\rho$ risk functional associated with ρ.

For law-invariant ρ and $\mathbb{F}(\mathcal{X}) := \{F_X : X \in \mathcal{X}\}$, we may define a map

$$\mathcal{R}_\rho : \mathbb{F}(\mathcal{X}) \longrightarrow \mathbb{R} \qquad \text{by} \qquad \mathcal{R}_\rho(F) \; := \; \rho(X_F),$$

where $X_F \in \mathcal{X}$ has df F. We call $\mathcal{R}_\rho$ risk functional associated with ρ.

2. Estimation in the context of aggregate risks

Consider a “homogeneous” insurance collective

- $X_1, \ldots, X_n \sim \mu$ i.i.d. individual claims in the next insurance period, $\sum_{i=1}^n X_i \sim \mu^{*n}$ total claim in the next insurance period, individual claim distribution μ be unknown
- $Y_1, \ldots, Y_{u_n} \sim \mu$ i.i.d. indiv. claims in the previous insurance period, $u_n/n \longrightarrow c \in (0, \infty)$

Goal

- information on the premium $\rho(\sum_{i=1}^n X_i) = \mathcal{R}_\rho(\mu^{*n})$ for the whole collective

Method

- choose reasonable estimator $\widehat{\mu}_{u_n}^{*n}$ for μ^{*n} based on $Y_1, \ldots, Y_{u_n}$
- use $\mathcal{R}_\rho(\widehat{\mu}_{u_n}^{*n})$ as estimator for $\mathcal{R}_\rho(\mu^{*n})$

Questions

1. What is a reasonable estimator $\widehat{\mu}^{*n}_{u_n}$ for μ^{*n} ?
2. Simple representation of the (estimated) indiv. premium $\frac{1}{n}\mathcal{R}_\rho(\widehat{\mu}^{*n}_{u_n})$?
3. What can be said about the relative error $\frac{1}{n}\big(\mathcal{R}_\rho(\widehat{\mu}^{*n}_{u_n}) - \mathcal{R}_\rho(\mu^{*n})\big)$?

Questions

1. What is a reasonable estimator $\widehat{\mu}_{u_n}^{*n}$ for μ^{*n} ?

Answer

The Central Limit Theorem and the Glivenko–Cantelli Theorem suggest

$$\widehat{\mu}_{u_n}^{*n} := \mathcal{N}_{n\widehat{m}_{u_n}, n\widehat{s}_{u_n}^2} \qquad \text{and} \qquad \widehat{\mu}_{u_n}^{*n} := (\widehat{\mu}_{u_n})^{*n},$$

respectively, where

$$\begin{aligned} \widehat{m}_{u_n} &= \text{empirical mean of } Y_1, \ldots, Y_{u_n} \\ \widehat{s}_{u_n} &= \text{empirical standard deviation of } Y_1, \ldots, Y_{u_n} \\ \widehat{\mu}_{u_n} &= \text{empirical measure of (uniform distribution on) } Y_1, \ldots, Y_{u_n}. \end{aligned}$$

Notice that the empirical measure $\widehat{\mu}_{u_n}$ can be calculated by the Panjer recursion when $\mu[\{0\}] > 0$.

Questions

2. Simple representation of the (estimated) indiv. premium $\frac{1}{n}\mathcal{R}_\rho(\widehat{\mu}_{u_n}^{*n})$?

Answer

For every law-invariant coherent risk measure ρ on $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbb{P})$ and every $\mu \in \mathcal{M}(\mathcal{X} \cap L^\lambda(\Omega, \mathcal{F}, \mathbb{P}))$ with $\lambda = \lambda(p) > 2$:

$$\begin{aligned}
\frac{1}{n}\mathcal{R}_\rho\big(\mu^{*n}\big) &= m + \frac{\mathcal{R}_\rho(\mathcal{N}_{0,1})}{\sqrt{n}}\, s + \mathcal{O}(n^{-1/2-\gamma}) \\
\frac{1}{n}\mathcal{R}_\rho\big(\mathcal{N}_{n\widehat{m}_{u_n},\, n\widehat{s}^2_{u_n}}\big) &= \widehat{m}_{u_n} + \frac{\mathcal{R}_\rho(\mathcal{N}_{0,1})}{\sqrt{n}}\, \widehat{s}_{u_n} \\
\frac{1}{n}\mathcal{R}_\rho\big((\widehat{\mu}_{u_n})^{*n}\big) &= \widehat{m}_{u_n} + \frac{\mathcal{R}_\rho(\mathcal{N}_{0,1})}{\sqrt{n}}\, \widehat{s}_{u_n} + \mathcal{O}_{\mathbb{P}\text{-a.s.}}(n^{-1/2-\gamma})
\end{aligned}$$

Krätschmer/H. Z. (2011)
Lauer/H. Z. (2014)

$$\gamma := \min\{\lambda - 2; 1\}/2$$

Questions

2. Simple representation of the (estimated) indiv. premium $\frac{1}{n}\mathcal{R}_\rho(\widehat{\mu}_{u_n}^{*n})$?

Answer

For instance, the first identity follows from

$$\begin{aligned}
\mathcal{R}_\rho(\mu^{*n}) &= \mathcal{R}_\rho(\mathcal{N}_{nm,\,ns^2}) + \big(\mathcal{R}_\rho(\mu^{*n}) - \mathcal{R}_\rho(\mathcal{N}_{nm,\,ns^2})\big) \\
&= \rho(nm + \sqrt{n}sZ) + \big(\rho(\sqrt{n}sZ_n + nm) - \rho(\sqrt{n}sZ + nm)\big) \\
&= nm + \sqrt{n}s\rho(Z) + \sqrt{n}s\big(\rho(Z_n) - \rho(Z)\big) \\
&= nm + \sqrt{n}s\mathcal{R}_\rho(\mathcal{N}_{0,1}) + \sqrt{n}s\big(\mathcal{R}_\rho(\mathrm{law}\{Z_n\}) - \mathcal{R}_\rho(\mathcal{N}_{0,1})\big)
\end{aligned}$$

$\big($with $Z_n := \frac{1}{\sqrt{n}s}\sum_{i=1}^n (X_i - m)$ and $Z \sim \mathcal{N}_{0,1}\big)$ and

$$\begin{aligned}
&\sqrt{n}s\,\big|\mathcal{R}_\rho(\mathrm{law}\{Z_n\}) - \mathcal{R}_\rho(\mathcal{N}_{0,1})\big| \\
&\quad\leq \sqrt{n}s \cdot \mathrm{const}_\rho \cdot \sup_{x\in\mathbb{R}} |F_{Z_n}(x) - \Phi_{0,1}(x)|\,(1 + |x|^\lambda) \\
&\quad\leq \sqrt{n}s \cdot \mathrm{const}_\rho \cdot \mathrm{const}_\lambda \cdot n^{-\gamma}
\end{aligned}$$

Questions

3. What can be said about the relative error $\frac{1}{n}\big(\mathcal{R}_\rho(\widehat{\mu}_{u_n}^{*n}) - \mathcal{R}_\rho(\mu^{*n})\big)$?

Answer

For every law-invariant coherent risk measure ρ on $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbb{P})$ and every $\mu \in \mathcal{M}(\mathcal{X} \cap L^\lambda(\Omega, \mathcal{F}, \mathbb{P}))$ with $\lambda = \lambda(p) > 2$:

$$\begin{aligned}
\frac{1}{n}\big(\mathcal{R}_\rho(\mathcal{N}_{nm,\,ns^2}) - \mathcal{R}_\rho(\mu^{*n})\big) &= \mathcal{O}(n^{-1}),\\
\frac{1}{n}\big(\mathcal{R}_\rho(\mathcal{N}_{n\widehat{m}_{u_n},\,n\widehat{s}^2_{u_n}}) - \mathcal{R}_\rho(\mu^{*n})\big) &= (\widehat{m}_{u_n} - m) + o_{\mathbb{P}\text{-a.s.}}(n^{-1/2}),\\
\frac{1}{n}\big(\mathcal{R}_\rho((\widehat{\mu}_{u_n})^{*n}) - \mathcal{R}_\rho(\mu^{*n})\big) &= (\widehat{m}_{u_n} - m) + o_{\mathbb{P}\text{-a.s.}}(n^{-1/2}).
\end{aligned}$$

Krätschmer/H. Z. (2011)
Lauer/H. Z. (2014)

$\gamma := \min\{\lambda - 2; 1\}/2$

Questions

3. What can be said about the relative error $\frac{1}{n}\big(\mathcal{R}_\rho(\widehat{\mu}_{u_n}^{*n}) - \mathcal{R}_\rho(\mu^{*n})\big)$?

Answer

Note that

$$
\begin{aligned}
& n^r\Big((\widehat{m}_{u_n} - m) + o_{\mathbb{P}\text{-a.s.}}(n^{-1/2})\Big) \\
& \quad = \quad \frac{u_n^r}{n^r} \cdot u_n^r(\widehat{m}_{u_n} - m) + \frac{o_{\mathbb{P}\text{-a.s.}}(n^{-1/2})}{n^{-r}} \\
& \quad \xrightarrow{\text{a.s.}} \quad 0 \qquad \text{for all } r < 1/2
\end{aligned}
$$

and

$$
\begin{aligned}
& \sqrt{u_n}\Big((\widehat{m}_{u_n} - m) + o_{\mathbb{P}\text{-a.s.}}(n^{-1/2})\Big) \\
& \quad = \quad \sqrt{u_n}(\widehat{m}_{u_n} - m) + \frac{o_{\mathbb{P}\text{-a.s.}}(n^{-1/2})}{u_n^{-1/2}} \\
& \quad \xrightarrow{\text{d}} \quad Z \sim \mathcal{N}_{0,s^2}
\end{aligned}
$$

Questions

3. What can be said about the relative error $\frac{1}{n}\big(\mathcal{R}_\rho(\widehat{\mu}_{u_n}^{*n}) - \mathcal{R}_\rho(\mu^{*n})\big)$?

Answer

. . . in particular,

$$\left[\frac{\mathcal{R}_\rho\big(\widehat{\mu}_{u_n}^{*n}\big)}{n} - \frac{\widehat{s}_{u_n}}{\sqrt{u_n}}\,\Phi^{-1}\Big(1-\frac{\alpha}{2}\Big)\,,\;\frac{\mathcal{R}_\rho\big(\widehat{\mu}_{u_n}^{*n}\big)}{n} - \frac{\widehat{s}_{u_n}}{\sqrt{u_n}}\,\Phi^{-1}\Big(\frac{\alpha}{2}\Big)\right]$$

provides an asymptotic confidence interval for the individual premium $\frac{1}{n}\mathcal{R}_\rho(\mu^{*n})$ at level α for both $\widehat{\mu}_{u_n}^{*n} := \mathcal{N}_{n\widehat{m}_{u_n},n\widehat{s}^2_{u_n}}$ and $\widehat{\mu}_{u_n}^{*n} := (\widehat{\mu}_{u_n})^{*n}$.

2. Estimation in the context of time series

Situation

- $X_1, \ldots, X_n \; (, X_{n+1}, \ldots)$ strictly stationary observations
- one-dimensional marginal distribution μ be unknown

Goal

- information on $\mathcal{R}_\rho(\mu)$

Method

- $\widehat{\mu}_n :=$ empirical measure of $X_1, \ldots, X_n$
- use $\mathcal{R}_\rho(\widehat{\mu}_n)$ as estimator for $\mathcal{R}_\rho(\mu)$

2. Estimation in the context of time series

Situation

- $X_1, \ldots, X_n \; (, X_{n+1}, \ldots)$ strictly stationary observations
- one-dimensional marginal distribution μ be unknown

Goal

- information on $\mathcal{R}_\rho(\mu)$

Questions

1. asymptotic distribution, bootstrap
2. consistency, a.s. rates
3. qualitative robustness

2. Estimation in the context of time series

2.1 Asymptotic distribution, bootstrap

Question

$$\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \xrightarrow{\mathsf{d}} \quad ?$$

FDM approach

1. show $\sqrt{n}(\widehat{F}_n - F) \xrightarrow{\mathsf{d}} B^\circ$
2. determine the "derivative" $\dot{\mathcal{R}}_{\rho,F}(\cdot)$
3. conclude $\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \xrightarrow{\mathsf{d}} \dot{\mathcal{R}}_{\rho,F}(B^\circ)$

Question

$$\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \overset{\mathsf{d}}{\longrightarrow} \quad ?$$

FDM approach ...

1. show $\sqrt{n}(\widehat{F}_n - F) \overset{\mathsf{d}}{\longrightarrow} B^\circ$
2. determine the "derivative" $\dot{\mathcal{R}}_{\rho,F}(\cdot)$
3. conclude $\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \overset{\mathsf{d}}{\longrightarrow} \dot{\mathcal{R}}_{\rho,F}(B^\circ)$

... in conventional fashion

1. weak convergence in $D_{\mathsf{b}} = D_{\mathsf{b}}(\mathbb{R})$ w.r.t. sup-norm $\|\cdot\|_\infty$

Question

$$\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \xrightarrow{\mathsf{d}} \quad ?$$

FDM approach ...

1. show $\sqrt{n}(\widehat{F}_n - F) \xrightarrow{\mathsf{d}} B^\circ$
2. determine the "derivative" $\dot{\mathcal{R}}_{\rho,F}(\cdot)$
3. conclude $\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \xrightarrow{\mathsf{d}} \dot{\mathcal{R}}_{\rho,F}(B^\circ)$

... in conventional fashion

1. weak convergence in $D_{\mathsf{b}} = D_{\mathsf{b}}(\mathbb{R})$ w.r.t. sup-norm $\|\cdot\|_\infty$
2. Hadamard derivative w.r.t. sup-norm $\|\cdot\|_\infty$

Question

$$\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \overset{\mathsf{d}}{\longrightarrow} \quad ?$$

FDM approach ...

1. show $\sqrt{n}(\widehat{F}_n - F) \overset{\mathsf{d}}{\longrightarrow} B^\circ$
2. determine the "derivative" $\dot{\mathcal{R}}_{\rho,F}(\cdot)$
3. conclude $\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \overset{\mathsf{d}}{\longrightarrow} \dot{\mathcal{R}}_{\rho,F}(B^\circ)$

... in conventional fashion

1. weak convergence in $D_{\mathsf{b}} = D_{\mathsf{b}}(\mathbb{R})$ w.r.t. sup-norm $\|\cdot\|_\infty$
2. Hadamard derivative w.r.t. sup-norm $\|\cdot\|_\infty$
3. by means of the classical FDM (Gill 1989, etc.)

Question

$$\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \stackrel{\mathsf{d}}{\longrightarrow} \quad ?$$

FDM approach ...

1. show $\sqrt{n}(\widehat{F}_n - F) \stackrel{\mathsf{d}}{\longrightarrow} B^\circ$
2. determine the "derivative" $\dot{\mathcal{R}}_{\rho,F}(\cdot)$
3. conclude $\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \stackrel{\mathsf{d}}{\longrightarrow} \dot{\mathcal{R}}_{\rho,F}(B^\circ)$

... in conventional fashion ...

1. weak convergence in $D_{\mathsf{b}} = D_{\mathsf{b}}(\mathbb{R})$ w.r.t. sup-norm $\|\cdot\|_\infty$
2. Hadamard derivative w.r.t. sup-norm $\|\cdot\|_\infty$
3. by means of the classical FDM (Gill 1989, etc.)

... does <u>not</u> cover any law-invariant <u>coherent</u> risk measure ρ

Question

$$\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \stackrel{\mathsf{d}}{\longrightarrow} \quad ?$$

FDM approach ...

1. show $\sqrt{n}(\widehat{F}_n - F) \stackrel{\mathsf{d}}{\longrightarrow} B^\circ$
2. determine the "derivative" $\dot{\mathcal{R}}_{\rho,F}(\cdot)$
3. conclude $\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \stackrel{\mathsf{d}}{\longrightarrow} \dot{\mathcal{R}}_{\rho,F}(B^\circ)$

... in new fashion ...

1. weak convergence in $D_\phi = D_\phi(\mathbb{R})$ w.r.t. weighted sup-norm $\|\cdot\|_\phi$
2. quasi-Hadamard derivative w.r.t. weighted sup-norm $\|\cdot\|_\phi$
3. by means of the modified FDM (Beutner/H. Z. 2010)

... covers "every" law-invariant coherent risk measure ρ

Functional Delta-Methode (informally)

$$\begin{aligned}
\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F) \quad &= \quad \dot{\mathcal{R}}_{\rho,F}(\widehat{F}_n - F) + \text{``rest''} \\
\sqrt{n}(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)) \quad &= \quad \dot{\mathcal{R}}_{\rho,F}\big(\sqrt{n}(\widehat{F}_n - F)\big) + \sqrt{n}\ \text{``rest''} \\
&\longrightarrow \quad \dot{\mathcal{R}}_{\rho,F}(B^\circ) \qquad\qquad + \ 0
\end{aligned}$$

$\mathcal{R}_\rho : \mathbb{F}(\mathcal{X}) \longrightarrow \mathbb{R}$, $\quad \mathbb{F}(\mathcal{X}) \subset D_{\mathsf{b}}$, $\quad \|\cdot\|_\infty$ sup-norm on D_{b}

Functional Delta-Methode (informally)

$$\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F) \quad = \quad \dot{\mathcal{R}}_{\rho,F}(\widehat{F}_n - F) + \text{“rest”}$$

$$\begin{aligned} \sqrt{n}(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)) \quad &= \quad \dot{\mathcal{R}}_{\rho,F}\big(\sqrt{n}(\widehat{F}_n - F)\big) + \sqrt{n}\,\text{“rest”} \\ &\longrightarrow \quad \dot{\mathcal{R}}_{\rho,F}(B^\circ) + 0 \end{aligned}$$

What kind of “derivative” is expedient?

What kind of "derivative" is expedient?

$\mathcal{R}_\rho : \mathbb{F}(\mathcal{X}) \longrightarrow \mathbb{R}$ Hadamard diff.ble at F if:

There is some $\dot{\mathcal{R}}_{\rho,F} \in L(D_\mathsf{b}, \mathbb{R})$ such that for each $v \in D_\mathsf{b}$

$$\lim_{n\to\infty} \left| \dot{\mathcal{R}}_{\rho,F}(v) - \frac{\mathcal{R}_\rho(F + h_n v_n) - \mathcal{R}_\rho(F)}{h_n} \right| = 0$$

for all

- $(h_n) \subset \mathbb{R}_{++}$ mit $h_n \to 0$,
- $(v_n) \subset D_\mathsf{b}$ with $\|v - v_n\|_\infty \to 0$

What kind of "derivative" is expedient?

$\mathcal{R}_\rho : \mathbb{F}(\mathcal{X}) \longrightarrow \mathbb{R}$ Hadamard diff.ble at F if:

There is some $\dot{\mathcal{R}}_{\rho,F} \in L(D_\mathsf{b}, \mathbb{R})$ such that for each $v \in D_\mathsf{b}$

$$\lim_{n\to\infty} \left| \dot{\mathcal{R}}_{\rho,F}\,(v) - \frac{\mathcal{R}_\rho(F + h_n v_n) - \mathcal{R}_\rho(F)}{h_n} \right| = 0$$

for all

- $(h_n) \subset \mathbb{R}_{++}$ mit $h_n \to 0$, $\quad h_n := n^{-1/2}$
- $(v_n) \subset D_\mathsf{b}$ with $\|v - v_n\|_\infty \to 0$ $\quad v_n := \sqrt{n}(\widehat{F}_n - F), \quad v = B^\circ$

What kind of "derivative" is expedient?

$\mathcal{R}_\rho : \mathbb{F}(\mathcal{X}) \longrightarrow \mathbb{R}$ Hadamard diff.ble at F tangentially to C_b if:

There is some $\dot{\mathcal{R}}_{\rho,F} \in L(C_\mathsf{b}, \mathbb{R})$ such that for each $v \in C_\mathsf{b}$

$$\lim_{n\to\infty} \left| \dot{\mathcal{R}}_{\rho,F}\,(v) - \frac{\mathcal{R}_\rho(F + h_n v_n) - \mathcal{R}_\rho(F)}{h_n} \right| = 0$$

for all

- $(h_n) \subset \mathbb{R}_{++}$ mit $h_n \to 0$, $\quad h_n := n^{-1/2}$
- $(v_n) \subset D_\mathsf{b}$ with $\|v - v_n\|_\infty \to 0$ $\quad v_n := \sqrt{n}(\widehat{F}_n - F), \quad v = B^\circ$

What kind of "derivative" is expedient?

$\mathcal{R}_\rho : \mathbb{F}(\mathcal{X}) \longrightarrow \mathbb{R}$ Hadamard diff.ble at F tangentially to C_b if:

There is some $\dot{\mathcal{R}}_{\rho,F} \in L(C_\mathsf{b}, \mathbb{R})$ such that for each $v \in C_\mathsf{b}$

$$\lim_{n\to\infty} \left| \dot{\mathcal{R}}_{\rho,F}\,(v) - \frac{\mathcal{R}_\rho(F + h_n v_n) - \mathcal{R}_\rho(F)}{h_n} \right| = 0$$

for all

- $(h_n) \subset \mathbb{R}_{++}$ mit $h_n \to 0$,
- $(v_n) \subset D_\mathsf{b}$ with $\|v - v_n\|_\infty \to 0$

But for coherent ρ there are "too many" (v_n) !!!

What kind of "derivative" is expedient?

$\mathcal{R}_\rho : \mathbb{F}(\mathcal{X}) \longrightarrow \mathbb{R}$ **quasi**-Hadam. diff.ble at F tangentially to $C_\phi\langle D_\phi\rangle$ if:

There is some $\dot{\mathcal{R}}_{\rho,F} \in L(C_\phi, \mathbb{R})$ such that for each $v \in C_\phi$

$$\lim_{n\to\infty} \left| \dot{\mathcal{R}}_{\rho,F}(v) - \frac{\mathcal{R}_\rho(F + h_n v_n) - \mathcal{R}_\rho(F)}{h_n} \right| = 0$$

for all

- $(h_n) \subset \mathbb{R}_{++}$ mit $h_n \to 0$,
- $(v_n) \subset D_\phi$ with $\|v - v_n\|_\phi \to 0$ Beutner/H. Z. 2010

What kind of "derivative" is expedient?

$\mathcal{R}_\rho : \mathbb{F}(\mathcal{X}) \longrightarrow \mathbb{R}$ **quasi**-Hadam. diff.ble at F tangentially to $C_\phi\langle D_\phi\rangle$ if:

There is some continuous $\dot{\mathcal{R}}_{\rho,F} : C_\phi \longrightarrow \mathbb{R}$ such that for each $v \in C_\phi$

$$\lim_{n\to\infty} \left| \dot{\mathcal{R}}_{\rho,F}(v) - \frac{\mathcal{R}_\rho(F + h_n v_n) - \mathcal{R}_\rho(F)}{h_n} \right| = 0$$

for all

- $(h_n) \subset \mathbb{R}_{++}$ mit $h_n \to 0$,
- $(v_n) \subset D_\phi$ with $\|v - v_n\|_\phi \to 0$ Beutner/H. Z. 2010

What kind of "derivative" is expedient?

$\mathcal{R}_\rho : \mathbb{F}(\mathcal{X}) \longrightarrow \mathbb{R}$ **quasi**-Hadam. diff.ble at F tangentially to $C_\phi \langle D_\phi \rangle$ if:

There is some continuous $\dot{\mathcal{R}}_{\rho,F} : C_\phi \longrightarrow \mathbb{R}$ such that for each $v \in C_\phi$

$$\lim_{n\to\infty} \left| \dot{\mathcal{R}}_{\rho,F}(v) - \frac{\mathcal{R}_\rho(F + h_n v_n) - \mathcal{R}_\rho(F)}{h_n} \right| = 0$$

for all

- $(h_n) \subset \mathbb{R}_{++}$ mit $h_n \to 0$, $\quad h_n := n^{-1/2}$
- $(v_n) \subset D_\phi$ with $\|v - v_n\|_\phi \to 0$ $\quad v_n := \sqrt{n}(\widehat{F}_n - F), \quad v = B^\circ$

What kind of "derivative" is expedient?

Example

$\mathcal{R}_\rho(F) := \int x\, dF(x)$

- $\dot{\mathcal{R}}_{\rho,F}(v) \;:=\; -\int v(x)\, dx, \quad v \in C_{\mathsf{b}}$
- $(v_n) \subset D_{\mathsf{b}}$

But $\|v - v_n\|_\infty \to 0$ does *not* imply

$$\left|\dot{\mathcal{R}}_{\rho,F}(v) - \frac{\mathcal{R}_\rho(F + h_n v_n) - \mathcal{R}_\rho(F)}{h_n}\right| = \left|\int \big(v(x) - v_n(x)\big)\, dx\right| \to 0.$$

What kind of "derivative" is expedient?

Example

$\mathcal{R}_\rho(F) := \int x\, dF(x)$

- $\dot{\mathcal{R}}_{\rho,F}(v) \;:=\; -\int v(x)\, dx, \quad v \in C_\phi$
- $(v_n) \subset D_\phi, \quad \int 1/\phi(x)\, dx < \infty$

~~But~~ $\|v - v_n\|_\phi \to 0$ does ~~*not*~~ imply

$$\left|\dot{\mathcal{R}}_{\rho,F}(v) - \frac{\mathcal{R}_\rho(F + h_n v_n) - \mathcal{R}_\rho(F)}{h_n}\right| = \left|\int \big(v(x) - v_n(x)\big)\, dx\right| \to 0.$$

To do

1. show $\sqrt{n}(\widehat{F}_n - F) \xrightarrow{\mathsf{d}} B^\circ$ in D_ϕ w.r.t. $\|\cdot\|_\phi$
2. determine quasi-Hadamard derivative $\dot{\mathcal{R}}_{\rho,F}(\cdot)$ of $\mathcal{R}_\rho$ w.r.t. $\|\cdot\|_\phi$

To do

1. show $\sqrt{n}(\widehat{F}_n - F) \xrightarrow{\mathsf{d}} B^\circ$ in D_ϕ w.r.t. $\|\cdot\|_\phi$
2. determine quasi-Hadamard derivative $\dot{\mathcal{R}}_{\rho,F}(\cdot)$ of $\mathcal{R}_\rho$ w.r.t. $\|\cdot\|_\phi$

Results on 1.

independent	Shorack/Wellner (1986)	
α-mixing	Shao/Yu (1996)	
β-mixing	Arcones/Yu (1994)	ARMA, GARCH, ...
ρ-mixing	Shao/Yu (1996)	
associated	Shao/Yu (1996)	
long-memory	Beutner/Wu/H. Z. (2012)	ARFIMA, ...
	Buchsteiner (2014)	
	($\sqrt{n}$ has to be replaced by data dependent rate!)	

To do

1. show $\sqrt{n}(\widehat{F}_n - F) \xrightarrow{\mathsf{d}} B^\circ$ in D_ϕ w.r.t. $\|\cdot\|_\phi$
2. determine quasi-Hadamard derivative $\dot{\mathcal{R}}_{\rho,F}(\cdot)$ of $\mathcal{R}_\rho$ w.r.t. $\|\cdot\|_\phi$

Results on 1.

The limit in distribution B° is a centered Gaussian process B° with covariance function ...

$\Gamma(x_0, x_1) = F(x_0 \wedge x_1)(1 - F)(x_0 \vee x_1)$ for independent data

$\Gamma(x_0, x_1) = F(x_0 \wedge x_1)(1 - F(x_0 \vee x_1))$ for weakly depend. data

$+ \sum_{i=0}^{1} \sum_{k=2}^{\infty} \mathbb{C}\mathrm{ov}(\mathbb{1}_{\{X_1 \leq x_i\}}, \mathbb{1}_{\{X_k \leq x_{1-i}\}})$

To do

1. show $\sqrt{n}(\widehat{F}_n - F) \xrightarrow{\text{d}} B^\circ$ in D_ϕ w.r.t. $\|\cdot\|_\phi$
2. determine quasi-Hadamard derivative $\dot{\mathcal{R}}_{\rho,F}(\cdot)$ of $\mathcal{R}_\rho$ w.r.t. $\|\cdot\|_\phi$

Results on 2.

Quasi-Hadamard differentiability of $\mathcal{R}_\rho$ at F tangentially to $C_\phi\langle D_\phi\rangle$ means that there is some continuous map $\dot{\mathcal{R}}_\rho : C_\phi \longrightarrow \mathbb{R}$ such that

$$\lim_{n\to\infty} \left|\frac{\mathcal{R}_\rho(F + h_n v_n) - \mathcal{R}_\rho(F)}{h_n} - \dot{\mathcal{R}}_{\rho,F}(v)\right| = 0$$

holds for every triplet $(v, (v_n), (h_n)) \in C_\phi \times D_\phi^{\mathbb{N}} \times (0, \infty)^{\mathbb{N}}$ satisfying $\|v_n - v\|_\phi \to 0$, $(F + h_n v_n) \subset \mathbb{F}(\mathcal{X})$, and $h_n \to 0$.

To do

1. show $\sqrt{n}(\widehat{F}_n - F) \xrightarrow{\mathsf{d}} B^\circ$ in D_ϕ w.r.t. $\|\cdot\|_\phi$
2. determine quasi-Hadamard derivative $\dot{\mathcal{R}}_{\rho,F}(\cdot)$ of $\mathcal{R}_\rho$ w.r.t. $\|\cdot\|_\phi$

Results on 2.

For every law-invariant coherent risk measure ρ on $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbb{P})$ (for some $p \in [1, \infty]$) and every $F \in \mathbb{F}(\mathcal{X})$ satisfying mild conditions, the risk functional $\mathcal{R}_\rho$ is quasi-Hadamard differentiable at F tangentially to $C_\phi\langle D_\phi\rangle$ with

$$\dot{\mathcal{R}}_{\rho,F}(v) \;=\; -\sup_{g\in\mathcal{G}_\rho}\int_{F^{\rightarrow}(0)}^{F^{\leftarrow}(1)} g'(F(x))\, v(x)\, dx$$

with $\mathcal{G}_\rho$ from the Kusuoka representation and ϕ depending on ρ and p.

Krätschmer/Schied/H. Z. (2013)

To do

1. show $\sqrt{n}(\widehat{F}_n - F) \xrightarrow{\text{d}} B^\circ$ in D_ϕ w.r.t. $\|\cdot\|_\phi$
2. determine quasi-Hadamard derivative $\dot{\mathcal{R}}_{\rho,F}(\cdot)$ of $\mathcal{R}_\rho$ w.r.t. $\|\cdot\|_\phi$

Results on 2.

For every coherent *distortion risk measure* ρ_g on $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbb{P})$ (for some $p \in [1, \infty]$) and every $F \in \mathbb{F}(\mathcal{X})$ satisfying mild conditions, the risk functional $\mathcal{R}_\rho$ is quasi-Hadamard differentiable at F tangentially to $C_\phi\langle D_\phi\rangle$ with

$$\dot{\mathcal{R}}_{\rho,F}(v) \;=\; -\int_{F^{\rightarrow}(0)}^{F^{\leftarrow}(1)} g'(F(x))\, v(x)\, dx$$

with ϕ depending on ρ_g and p.

Beutner/H. Z. (2010), Krätschmer/Schied/H. Z. (2013)

To do

1. show $\sqrt{n}(\widehat{F}_n - F) \xrightarrow{\mathsf{d}} B^\circ$ in D_ϕ w.r.t. $\|\cdot\|_\phi$
2. determine quasi-Hadamard derivative $\dot{\mathcal{R}}_{\rho,F}(\cdot)$ of $\mathcal{R}_\rho$ w.r.t. $\|\cdot\|_\phi$

Then

The limit in distribution of $\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big)$ is

$$\dot{\mathcal{R}}_{\rho,F}(B^\circ) \;=\; -\int_{F^{\rightarrow}(0)}^{F^{\leftarrow}(1)} g'(F(x))\, B^\circ(x)\, dx.$$

If B° is a mean zero Gaussian process with covariance function Γ, then

$$\dot{\mathcal{R}}_{\rho,F}(B^\circ) \;\sim\; \mathcal{N}_{0,s^2}$$

with

$$s^2 \;=\; s^2(F) \;:=\; \iint g'(F(x_0))\, \Gamma(x_0, x_1)\, g'(F(x_1))\, dx_0\, dx_1.$$

To do

1. show $\sqrt{n}(\widehat{F}_n - F) \xrightarrow{\text{d}} B^\circ$ in D_ϕ w.r.t. $\|\cdot\|_\phi$
2. determine quasi-Hadamard derivative $\dot{\mathcal{R}}_{\rho,F}(\cdot)$ of $\mathcal{R}_\rho$ w.r.t. $\|\cdot\|_\phi$

Then

The limit in distribution of $\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big)$ is

$$\dot{\mathcal{R}}_{\rho,F}(B^\circ) \;=\; -\int_{F^\rightarrow(0)}^{F^\leftarrow(1)} g'(F(x))\,B^\circ(x)\,dx.$$

If B° is a mean zero Gaussian process with covariance function Γ, then

$$\dot{\mathcal{R}}_{\rho,F}(B^\circ) \;\sim\; \mathcal{N}_{0,s^2}$$

with

$$s^2 \;=\; s^2(F) \;:=\; \iint g'(F(x_0))\,\Gamma(x_0,x_1)\,g'(F(x_1))\,dx_0\,dx_1.$$

Problem: The estimation of the asymptotic variance $s^2(F)$ often works only moderately well or is "impossible".

Way out: Bootstrapping the asymptotic distribution.

Beutner/H. Z. (2014) extended the *modified* FDM to the bootstrap, implying that for coherent distortion risk measures ρ:

$$\sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)\big) \overset{\mathsf{d}}{\approx} \sqrt{n}\big(\mathcal{R}_\rho(\widehat{F}_n^*) - \mathcal{R}_\rho(\widehat{F}_n)\big) \,|\, \widehat{F}_n, \qquad n \to \infty.$$

Here $\widehat{F}_n^*$ is a bootstrapped version of $\widehat{F}_n$ according to, for instance, ...

... Efron's bootstrap, the Bayesian bootstrap, or the wild bootstrap for independent data.

... the blockwise bootstrap for β-mixing data.

Efron's bootstrap

$$\widehat{F}_n^*(x;\omega,\widetilde{\omega}) := \frac{1}{n}\sum_{i=1}^{n} W_{n,i}(\widetilde{\omega})\mathbb{1}_{[X_i(\omega),\infty)}(x),$$

where $(W_{n,1},\ldots,W_{n,n})$ is multinomially distributed according to the parameters n and $p_1 = \cdots = p_n = 1/n$.

Informally, $\widehat{F}_n^*$ is the empirical distribution function of $X_{n,1}^*,\ldots,X_{n,n}^*$, where $X_{n,1}^*,\ldots,X_{n,n}^*$ are drawn (with replacement) from the "urn" $\{X_1,\ldots,X_n\}$.

2. Estimation in the context of time series

2.2 Consistency, a.s. rates

"Differentiability" implies "continuity" and even "Lipschitz continuity". This motivates the fact that it suffices to show

1. $\|\widehat{F}_n - F\|_\phi \xrightarrow{\text{a.s.}} 0$
2. $n^r\|\widehat{F}_n - F\|_\phi \xrightarrow{\text{a.s.}} 0$

in order to obtain respectively

1. $|\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)| \xrightarrow{\text{a.s.}} 0$
2. $n^r|\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)| \xrightarrow{\text{a.s.}} 0$

"Differentiability" implies "continuity" and even "Lipschitz continuity". This motivates the fact that it suffices to show

1. $\|\widehat{F}_n - F\|_\phi \xrightarrow{\text{a.s.}} 0$
2. $n^r\|\widehat{F}_n - F\|_\phi \xrightarrow{\text{a.s.}} 0$

in order to obtain respectively

1. $|\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)| \xrightarrow{\text{a.s.}} 0$
2. $n^r|\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F)| \xrightarrow{\text{a.s.}} 0$

Respective results:

1. For independent data: Andersen/Giné/Zinn (1988), ...
   For α-mixing data: H. Z. (2013)

2. For independent data: Andersen/Giné/Zinn (1988), ...
   For α-mixing data, $\phi \equiv 1$: H. Z. (2013), ...

2. Estimation in the context of time series

2.3 Qualitative robustness

Let ρ be a law-invariant coherent risk measure on $\mathcal{X} = L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $p \in [1, \infty]$, and equip $\mathcal{M}(\mathcal{X})$ with any metric d which generates the weak topology. For instance, $d = d_{\text{Lévy}}$ or $d = d_{\text{Proh}}$.

- $\mathbb{P}^\mu := \mu^{\otimes \mathbb{N}}$ for $\mu \in \mathcal{M}(\mathcal{X})$.
- $X_1, X_2, \ldots$ coordinate projections on $\mathbb{R}^{\mathbb{N}}$.
- $\widehat{\mathcal{R}}_n := \mathcal{R}_\rho(\widehat{\mu}_n)$ with $\widehat{\mu}_n$ the empirical measure of $X_1, \ldots, X_n$.

Definition

The sequence $(\widehat{\mathcal{R}}_n)$ is said to be robust at $\mathcal{M}_0 \subset \mathcal{M}(\mathcal{X})$ when for each $\mu_1 \in \mathcal{M}(\mathcal{X})$ and $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\mu_2 \in \mathcal{M}_0, \quad d(\mu_1, \mu_2) \le \delta \quad \Longrightarrow \quad d_{\text{Proh}}\big(\mathbb{P}^{\mu_1}_{\widehat{\mathcal{R}}_n}, \mathbb{P}^{\mu_2}_{\widehat{\mathcal{R}}_n}\big) \le \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Let ρ be a law-invariant coherent risk measure on $\mathcal{X} = L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $p \in [1, \infty]$, and equip $\mathcal{M}(\mathcal{X})$ with any metric d which generates the weak topology. For instance, $d = d_{\text{Lévy}}$ or $d = d_{\text{Proh}}$.

- $\mathbb{P}^\mu := \mu^{\otimes \mathbb{N}}$ for $\mu \in \mathcal{M}(\mathcal{X})$.
- $X_1, X_2, \ldots$ coordinate projections on $\mathbb{R}^\mathbb{N}$.
- $\widehat{\mathcal{R}}_n := \mathcal{R}_\rho(\widehat{\mu}_n)$ with $\widehat{\mu}_n$ the empirical measure of $X_1, \ldots, X_n$.

Definition

The sequence $(\widehat{\mathcal{R}}_n)$ is said to be robust at $\mathcal{M}_0 \subset \mathcal{M}(\mathcal{X})$ when for each $\mu_1 \in \mathcal{M}(\mathcal{X})$ and $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\mu_2 \in \mathcal{M}_0, \quad d(\mu_1, \mu_2) \leq \delta \quad \Longrightarrow \quad d_{\text{Proh}}\big(\mathbb{P}^{\mu_1}_{\widehat{\mathcal{R}}_n}, \mathbb{P}^{\mu_2}_{\widehat{\mathcal{R}}_n}\big) \leq \varepsilon \ \text{ for all } n \in \mathbb{N}.$$

Hampel's theorem

$$(\widehat{\mathcal{R}}_n) \text{ robust at } \mathcal{M}_0 \quad \Longleftrightarrow \quad \mathcal{R}_\rho|_{\mathcal{M}_0} \text{ continuous for weak topology}$$

Recall: p-weak topology

Let $p \in [0, \infty]$. On

$$\mathcal{M}_1^p(\mathbb{R}) := \left\{ \mu \in \mathcal{M}_1(\mathbb{R}) \, : \, \int |x|^p \, \mu(dx) < \infty \right\}$$

we may impose the p-weak topology, that is, the coarsest topology for which all mappings $\mu \longmapsto \int f \, d\mu$, $f \in C_p(\mathbb{R})$, are continuous, where

$$C_p(\mathbb{R}) := \left\{ f \in C(\mathbb{R}) \, : \, |f(x)| \leq c(1 + |x|^p) \text{ for some } c \in (0, \infty) \right\}.$$

Notice that

$$\begin{aligned} & \mu_n \to \mu \quad p\text{-weakly} \\ \iff \quad & \textstyle\int f \, d\mu_n \to \int f \, d\mu \text{ for all } f \in C_p(\mathbb{R}) \\ \iff \quad & \mu_n \to \mu \text{ weakly and } \textstyle\int |x|^p \, \mu_n(dx) \to \int |x|^p \, \mu(dx) \\ \iff \quad & d_{\mathrm{Proh}}(\mu_n, \mu) \to 0 \text{ and } \textstyle\int |x|^p \, \mu_n(dx) \to \int |x|^p \, \mu(dx) \end{aligned}$$

In original work on monetary risk measures the domain of $\rho : \mathcal{X} \longrightarrow \mathbb{R}$ was assumed to be $\mathcal{X} = L^\infty$. In view of

$$L^\infty \subset L^p \subset L^1 \subset L^0 \qquad (1 < p < \infty),$$

this is, of course, too restrictive for practical aspects. But:

Filipovic/Svindland (2012):

For every law-invariant coherent risk measure $\rho : L^\infty \longrightarrow \mathbb{R}$ there exists a unique ($\|\cdot\|_1$-lower-semicontinuous) law-invariant coherent risk measure

$$\overline{\rho} \, : \, L^1 \longrightarrow \mathbb{R} \cup \{+\infty\}$$

such that $\overline{\rho}$ is an extension of ρ.

- For the Average Value at Risk at level $\alpha \in (0,1)$

  $$\mathrm{AV@R}_\alpha(X) := \frac{1}{\alpha}\int_0^\alpha \mathrm{V@R}_s(X)\,ds$$

  the Filipovic–Svindland extension $\overline{\mathrm{AV@R}_\alpha}$ is finite on L^1.

- For the distortion risk measure associated with continuous convex distortion function g

  $$\rho_g(X) := -\int_{-\infty}^0 g(F_X(x))\,dx + \int_0^\infty \big(1 - g(F_X(x))\big)\,dx$$

  the Filipovic–Svindland extension $\overline{\rho_g}$ is finite on L^p iff

  $\int_0^1 g'_+(t)^{\frac{p}{p-1}}\,dt < \infty; \quad p > 1,$
  g'_+ is bounded; $\quad p = 1.$

- For the risk measure based on one-sided pth moments ($a \in [0,1]$)

  $$\rho_{p,a}(X) := \mathbb{E}[X] + a\,\mathbb{E}\big[((X - \mathbb{E}[X])^+)^p\big]^{1/p}$$

  the Filipovic–Svindland extension $\overline{\rho_{p,a}}$ is finite on L^q iff $q \geq p$.

Theorem (Krätschmer/Schied/H. Z. (2014))

For every law-invariant coherent risk measure $\rho : L^\infty \longrightarrow \mathbb{R}$ and $1 \leq p < \infty$ the following four statements are equivalent:

(i) $\overline{\rho}$ is finite on L^p.

(ii) $\mathcal{R}_{\overline{\rho}}$ is (finite and) p-weakly continuous on $\mathcal{M}(L^p)$.

(iii) $\mathcal{R}_\rho$ is p-weakly continuous on $\mathcal{M}(L^\infty)$.

(iv) ρ is continuous w.r.t. $\|\cdot\|_p$ at $X \equiv 0$.

Consequence

For a law-invariant coherent risk measure ρ with "natural" domain L^p ($p \in (1, \infty]$) the associated risk functional $\mathcal{R}_\rho$ is *not* weakly continuous but only p-weakly continuous.

Consequence

For a law-invariant coherent risk measure ρ with "natural" domain L^p ($p \in (1, \infty]$) the associated risk functional $\mathcal{R}_\rho$ is *not* weakly continuous but only p-weakly continuous.

In particular

For a law-invariant coherent risk measure ρ with "natural" domain L^p the sequence $(\widehat{\mathcal{R}}_n)$ is robust on a set $\mathcal{M}_0 \subset \mathcal{M}(L^p)$ if and only if the weak topology and the p-weak topology on $\mathcal{M}_0$ coincide.

Consequence

For a law-invariant coherent risk measure ρ with "natural" domain L^p ($p \in (1, \infty]$) the associated risk functional $\mathcal{R}_\rho$ is *not* weakly continuous but only p-weakly continuous.

In particular

For a law-invariant coherent risk measure ρ with "natural" domain L^p the sequence $(\widehat{\mathcal{R}}_n)$ is robust on a set $\mathcal{M}_0 \subset \mathcal{M}(L^p)$ if and only if the weak topology and the p-weak topology on $\mathcal{M}_0$ coincide.

Note

The following statements are equivalent:

(i) The weak and the p-weak topology on $\mathcal{M}_0$ coincide.

(ii) $\mathcal{M}_0$ is locally uniformly $|\cdot|^p$-integrating:
For each $\mu_1 \in \mathcal{M}_0$ and $\varepsilon > 0$ there are $\delta > 0$ and $a_0 > 0$ such that

$$\mu_2 \in \mathcal{M}_0,\ d(\mu_1, \mu_2) \le \delta \implies \int |x|^p \, \mathbb{1}_{\{|x|^p \ge a\}} \, \mu_2(dx) \le \varepsilon \text{ for } a \ge a_0.$$

Consequence

For a law-invariant coherent risk measure ρ with "natural" domain L^p ($p \in (1, \infty]$) the associated risk functional $\mathcal{R}_\rho$ is *not* weakly continuous but only p-weakly continuous.

In particular

For a law-invariant coherent risk measure ρ with "natural" domain L^p the sequence $(\widehat{\mathcal{R}}_n)$ is robust on a set $\mathcal{M}_0 \subset \mathcal{M}(L^p)$ if and only if the weak topology and the p-weak topology on $\mathcal{M}_0$ coincide.

That is

For a law-invariant coherent risk measure ρ with "natural" domain L^p:

The smaller p is, the larger are the sets on which the sequence $(\widehat{\mathcal{R}}_n)$ is robust!

Thank you!

Some references

- Beutner, E. and Zähle, H. (2010).
  A modified functional delta method and its application to the estimation of risk functionals. *Journal of Multivariate Analysis*, 101, 2452–2463.
- Beutner, E. and Zähle, H. (2014).
  A Functional Delta-Method for the bootstrap of quasi-Hadamard differentiable functionals, *Working Paper*.
- Beutner, E., Wu., W.B., and Zähle, H. (2012).
  Asymptotics for statistical functionals of long-memory sequences. *Stochastic Processes and their Applications*, 122, 910–929.
- Krätschmer, V. and Zähle, H. (2011).
  Sensitivity of risk measures with respect to the normal approximation of total claim distributions. *Insurance: Mathematics and Economics*, 49, 335–344.
- Krätschmer, V., Schied, A., and Zähle, H. (2014).
  Comparative and qualitative robustness of law-invariant risk measures. *Finance and Stochastics*, 18, 271–295.
- Krätschmer, V., Schied, A., and Zähle, H. (2014).
  Comparative and qualitative robustness of law-invariant risk measures, part II. *Working Paper*.
- Krätschmer, V., Schied, A., and Zähle, H. (2013).
  Quasi-Hadamard differentiability of general risk functionals and its application. *Submitted*.
- Lauer, A. and Zähle, H. (2014).
  Nonparametric estimation of risk measures of aggregate risks. *Working Paper*.
- Zähle, H. (2014).
  Marcinkiewicz–Zygmund and ordinary strong laws for empirical distribution functions and plug-in estimators. *Statistics*, 48, 951–964.